Toeholds and Takeovers

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Abstract

Part ownership of a takeover target can help a bidder win a takeover auction, often at a low price. A bidder with a "toehold" bids aggressively in a standard ascending auction because its offers are both bids for the remaining shares and asks for its own holdings. While the direct effect of a toehold on a bidder's strategy may be small, the indirect effect is large in a common value auction. When a firm bids more aggressively, its competitors face an increased winner's curse and must bid more conservatively. This allows the toeholder to bid more aggressively still, and so on. One implication is that a controlling minority shareholder may be immune to outside offers. The board of a target may increase the expected sale price by allowing a second bidder to buy a toehold on favorable terms, or by running a sealed bid auction.

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1 Introduction

Buying a stake or "toehold" in a takeover target is a common and profitable strategy.[1] The potential acquirer can gain either as a buyer who needs to pay a premium for fewer shares, or as a losing bidder who sells out at a profit. Therefore a company that owns a toehold has an incentive to bid aggressively, as every price it quotes represents not just a bid for the remaining shares but also an ask for its own holdings.

But this is the beginning of the story, not the end. Auctions of companies, at least when the bidders are "financial" buyers such as leveraged buyout firms rather than "strategic" buyers such as customers, suppliers, or competitors, are substantially common-value affairs. That is, differences in perceptions about the value of a company will often stem primarily from differences in expectations about the company's underlying business rather than differences in the expectations of different bidders of their ability to raise the value of the business.[2] The implication of common values is dramatic.

When a toehold makes a bidder more aggressive, it increases the winner's curse for a competitor. In a common value ascending auction, this will cause the competitor to bid more conservatively.[3] [4] The conservative competitor reduces the toeholder's winner's curse, allowing the toeholder to bid more aggressively still, and so on. The change in bidding strategies caused by a toehold will be much larger in a common value ascending auction than in a private value auction.

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[1]Well-known empirical studies that discuss toeholds include Bradley, Desai and Kim (1988), Franks and Harris (1989), Jarrell and Poulsen (1989), Stulz, Walkling, and Song (1990), Jennings and Mazzeo (1993), Betton and Eckbo (1997), Franks, Mayer and Renneboog (1997) and Jenkinson and Ljungqvist (1997). These studies indicate that a large percentage of bidders own toeholds, often of 10-20 percent or more, at the time they make offers. (Betton and Eckbo's highly comprehensive data set of 1353 takeover attempts shows that about half of the initial bidders have toeholds.) We know of no data on options granted to friendly bidders such as Kohlberg, Kravis, and Roberts in its offer for Borden or U.S. Steel in its offer for Marathon Oil, or similar devices which can effectively serve as "toehold substitutes". There is also little information on the differences in the types of bidders who acquire toeholds and those that do not.

[2]For non-controlling shareholders, stocks are almost entirely common value assets. For competing LBO groups, which are likely to apply similar managerial and financing techniques to acquired companies, the common value element probably dominates. When Wall Street analysts quote a company's "break-up value" they are essentially making common value estimates of the value of a company's businesses.

[3]In a private value ascending auction a non-toeholder will be unaffected by an opponent's bidding; if a competitor has a toehold then it will become more aggressive if it thinks there is less chance of its opponent dropping out at any given price.

[4]This observation has also been made by Bikhchandani (1988), in showing the value of a reputation for aggressive bidding in common value auctions.

Furthermore, it is not so much the change in the toeholder's own strategy that raises its profitability as it is the induced change in competitors' bidding that makes the toehold such an important strategic weapon. A bidder makes tradeoffs in deciding to become more aggressive, but unambiguously benefits from a competitor becoming more conservative.

While the intuition above is right when only one bidder has a toehold, things get more complex when two bidders have shares. Now each bidder wears both buyer and seller hats when quoting a price. A bidder who expects to lose the auction, and is primarily in selling mode, may quote a higher price against an opponent who has a large toehold and is therefore expected to be very aggressive. So in our model an exogenous increase in a bidder's toehold always increases its probability of winning and its expected profits, but sometimes increases the average price it pays when it wins. Our results are consistent with empirical findings that toeholds increase a bidder's chance of winning a takeover battle (Walkling (1985), Betton and Eckbo (1997)) but it is unclear whether they decrease (Jarrell and Poulsen (1989), Eckbo and Langhor (1989)), increase (Franks and Harris (1989)), or have no effect on (Stulz, Walkling, and Song (1990)) target returns. By contrast, the private-value models of Englebrecht-Wiggans (1994), Burkart (1995), and Singh (1994) imply that toeholds should unambiguously raise bids and prices, but that the effects should be relatively small. Only Hirshleifer (1995) concludes as we do that even a small toehold can have a large effect on the final price in a multiple-bidder takeover battle.[5]

Our model can explain why bidders sometimes seem to overpay for the companies they take over, without appealing to stories of managerial hubris or of management pursuing its own interests at the expense of shareholders. Here, bidding "too high" maximizes a bidders' ex-ante expected profits even though it sometimes loses money ex-post.[6]

The model also implies that an ownership stake of significantly less than 50% in a company may be sufficient to guarantee effective control; a toehold may make it much less

[5] Hirshleifer (1995, Section 4.5) shows that in the special case of full information, a small toehold can have a big effect on an ascending private-value auction. The firm with the lower value will drop out at a price just below the other bidder's valuation if it has a small toehold (and if any bidding costs are small enough), but if it has no toehold it will bid no further than its own valuation (and will withdraw from the bidding if there are any bidding costs).

[6] Burkart (1995) and Singh (1996) have made this point in the context of a private-value auction, but in their models a small toehold has only a small effect. Chowdhry and Nanda (1993) argue that an indebted firm may commit itself to aggressive bidding (and so sometimes deter competition) by committing to finance the acquisition through additional debt of equal or senior priority, and that this might sometimes lead to overpayment.

likely that an outside bidder will enter a takeover battle. This result is consistent with Walkling and Long (1984) and Jennings and Mazzeo (1993), who find that toeholds lower the probability of management resistence; of Stulz, Walkling and Song (1990), who report much larger toeholds in uncontested than in contested takeovers; and of Betton and Eckbo (1997), who find that greater toeholds increase the probability of a successful single-bid contest by lowering both the chance of entry by a rival bidder and target management resistance.[7]

Our analysis also makes predictions that have not yet been tested, because empirical work in the field has not distinguished between private-value and common-value auctions. Since a toehold should have a lesser effect on a private-value auction than a common-value auction, we believe that the incentive for acquiring a toehold is much lower for a "strategic" bidder than for a "financial" bidder. A financial bidder should generally not compete with a strategic bidder unless it has a toehold or other financial inducement.

Since a basic message of the analysis is that if just one bidder has a substantial toehold then that bidder can expect large profits, we consider two natural ways in which the management of the target company might seek to even the contest.

One approach is to replace a conventional ascending-bid takeover auction with a first-price auction in which bidders are permitted to make only a single sealed "best and final offer" and the company is sold at the highest bid.[8] Because a bidder's offer now affects the sale price only if the bidder wins, there is no incentive to bid up the price purely in order to "sell high". Therefore, with symmetric toeholds, bidders will be less aggressive in a first-price auction and prices will be lower on average in the first-price auction than in an ascending auction. However, with asymmetric toeholds the large toeholder being more aggressive in an ascending bid auction also means that the small toeholder becomes more

[7]Except that both Betton and Eckbo (1997) and Jennings and Mazzeo (1993) find that very small toeholds lead to *more* target management resistance than zero toeholds. This result would be explained if, as we argue next, financial bidders are more likely to acquire toeholds and, because they have no private-value advantage, are also more likely to be challenged.

[8]While it may be legally difficult for a board to refuse to consider higher subsequent offers, if it can award the highest sealed bidder a "breakup fee", options to buy stock, or options to purchase some of the company's divisions on favorable terms, then de facto it may create a first price auction. (A "break-up" fee is a fee that would be payable to the highest sealed bidder in the event that it did not ultimately win the company.) Thus our analysis can justify the use of "lock-up" provisions to support the credibility of a first-price auction. For previous analyses of the merits of allowing "lock-ups" see Kahan and Klausner (1996) and the references cited there.

conservative on average. Since it is the lower of the two bids that determines price, and the small toeholder is more likely to have the low bid, with small asymmetric toeholds prices will be lower on average in an ascending auction than in a first-price auction.

A second approach is to try to "level the playing field" by giving a second bidder the opportunity to acquire stock at a low price, narrowing the differences in toeholds. Doing so will make the auction for the company more competitive. While it would not pay to sell stock cheaply to two symmetric bidders, we show that the cost would be surprisingly small, because larger toeholds lead to more aggressive bidding. Therefore, the increased competition created by selling stock cheaply to only the smaller of two asymmetric bidders can easily swamp the "giveaway" aspect of such a deal. With small toeholds, it will always pay to subsidize the smaller toeholder in this way.

While our primary focus is on auctions of companies, there are several related problems to which our analysis can apply. Perhaps the most interesting at the moment is the sale of "stranded assets" by public utilities. In these sales of assets that are worth far less than book value, state public utilities commissions promise to reimburse utilities' shareholders for some percentage of the difference between the asset's sale price and the book value. If the percentage reimbursement is 80 percent, then the utility effectively has a toehold of 20 percent in the auctioned asset.[9] Other applications include the sharing of profits in bidding rings,[10] creditors' bidding in bankruptcy auctions,[11] and the negotiation of a partnership's dissolution. [12] More generally, the theory lends insight into problems in which a losing bidder cares how much the winner pays, as when a competitor in several auctions faces an aggregate budget constraint.[13]

There are two strands to the theoretical literature on toeholds. One strand, originated by Shleifer and Vishny (1986) and including Hirshleifer and Titman (1990) and Chowdhry and Jegadeesh (1994), focuses on the use of toeholds by a single bidder to combat the free

[9] That is, the utility is 20 cents better off if the asset is sold to someone else for a dollar more, and is only 80 cents worse off if it must bid an extra dollar to win the auction. This makes the utility's position strategically identical to a toeholder with a 20 percent stake in our model.

[10] See McAfee and McMillan (1992) and Engelbrecht-Wiggans (1994).

[11] See Burkart (1995).

[12] See Cramton, Gibbons, and Klemperer (1987).

[13] The theory here is also closely related to other examples in which one player has a small advantage (e.g. a small private-value advantage or a reputational advantage) in an otherwise pure common-value auction; see Bikhchandani (1988), Bulow and Klemperer (1997), and Klemperer (1997).

rider problem described by Grossman and Hart (1980). Owning a toehold gives a bidder a profit from a successful takeover, even if it has to pay the expected full value for any shares bought in a tender offer. While a larger toehold increases the chance that a tender offer will be successful, on average all of a bidder's profits will be accounted for by gains on the toehold. A larger toehold reduces the price a bidder will have to pay in the Shleifer and Vishny and Hirshleifer and Titman models, but increases it in the Chowdhry and Jegadeesh model.

The second strand focuses on bidding contests and assumes away the free rider problem. There are several justifications for this approach. The ability of a bidder that acquires a supermajority of the stock to force out non-tendering shareholders can eliminate the free-rider problem. Also, if small minority stakes can be left outstanding, the loss of liquidity in those shares can have the same effect in reducing their value as would measures that directly oppress minority investors, giving bidders an extra incentive to tender.

Engelbrecht-Wiggans (1994) has a private value model in which all bidders are symmetric and have identical toeholds. Burkart (1995) and Singh (1996) have private value models in which one bidder has a toehold and the other does not. In all these models a small toehold has only a small effect,[14] and a bidder with a toehold bids more aggressively so toeholds always raise prices. Of course, none of these models can show how a toehold can make a competitor more conservative, and so significantly raise a bidder's expected profits while lowering prices. In contrast to the free-rider models, in these models and ours bidders make profits beyond the direct gains on their toeholds.[15]

These private value models are probably most appropriate for auctions among "strategic" bidders whose differential valuations are not explained by varying perceptions about

[14]But in Hirshleifer's (1995) model without asymmetric information a small toehold has a large effect. See note 5.

[15]The free-rider models provide a theoretical foundation for the conventional wisdom that acquirers do not make profits on average, judged by their subsequent stock market performance. However, Loughran and Vijh (1996) show that acquirers who pay cash do make profits while those that issue stock underperform the market, just as other non-acquiring equity issuers do. So market prices may overstate the consideration paid in stock takeovers, and market returns may understate the real profitability of these transactions. (Similarly, Rau and Vermaelen (1996) show that "value" companies appear to make profits on tender offers, while "glamour" companies, those whose shares sell at a high multiple of book value, decline in the extended period following the issuance of new equity in a takeover.) These papers are therefore consistent with the "bidding contest" models of toeholds, including ours, in which bidders make profits. Of course, there are many non-public investors, such as private entrepreneurs and leveraged buyout firms, who make a business of acquiring and reorganizing companies, and appear to be very profitable on average.

what the target is worth on its own. However, we would predict that because toeholds are of much greater importance to "financial" bidders competing in common value auctions, toeholds are much more likely to be acquired by common-value bidders.

To focus clearly on the strategic effects we concentrate on the polar case of pure common values. Of course, in reality takeover targets have both private-value and common-value components, so our pure common-values model yields some results that are quantitatively implausible,[16] even though we believe they are qualitatively correct.[17] Our model also does not allow for the possibility of firms "jump-bidding", that is, discontinuously raising the bidding level to intimidate opponents into quitting the auction, as is often observed in practice. Jump-bidding is less likely when there are toeholds, since it is harder to discourage an opponent with a toehold from bidding, but would still arise if there were substantial bidding costs (including costs of entering the auction), especially with smaller toeholds and private-value components. Although we do not expect jump-bidding to affect our basic results and intuitions, it would probably attenuate their quantitative significance by making behavior closer to that in a first-price auction, so this is a further reason for not taking our results too literally when toeholds are small.[18]

Section 2 sets out our basic "common-values" model of two bidders who have toeholds in a target company, and also have private independent information about the value of that company. Were the bidders to completely share information, they would have the same valuation for the target.

Section 3 solves for the unique equilibrium of an ascending auction between the bidders.[19] Section 4 derives its properties and shows that asymmetric toeholds tend to lower sale prices.

Sections 5 and 6 discuss how the management of the target company might "change the

[16] For example, we find that bidders' probabilities of winning are in proportion to their toeholds even when the toeholds are arbitrarily small.

[17] It can be checked that the equilibrium we find is continuous as small private-value components are added. See also Bulow, Huang, and Klemperer (1995) for the general partially common-value, partially private-value, case.

[18] See Section 5 for our analysis of a first-price auction with toeholds, and see Avery (1996) and Daniel and Hirshleifer (1996), for pure common-value and pure private-value models, respectively, of jump-bidding in the absence of toeholds.

[19] Note that with toeholds we obtain a unique equilibrium in the ascending English auction even with pure common values. It is well-known that when bidders have no initial stakes in the object they are competing for, there is multiplicity of (perfect Bayesian) equilibria, but we show that (even arbitrarily small) toeholds resolve this multiplicity.

game" to reduce the advantage of the bidder with the larger toehold (or only toehold) and so raise the expected sale price. Section 5 solves and analyzes the equilibrium in a common value first price auction, while Section 6 considers the effect of offering stock cheaply or options to the bidder with the smaller toehold to make the auction more competitive. Section 7 extends our analysis to the case in which bidders' private signals are of different informativenesses, and shows that most of our results are unaffected.

Section 8 concludes.

2 The Model

Two risk-neutral bidders i and j compete to acquire a company. Bidder k ($k = i, j$) owns a share θ_k of the company, $0 < \theta_k < \frac{1}{2}$, and observes a private signal t_k. Bidders' shares are common knowledge[20] and exogenous.[21] Bidders' signals are independent, so without loss of generality we can normalise so that both the t_k are uniformly distributed on [0,1]. That is, a signal of $t_k = .23$ is more optimistic than 23% of the signals k might receive and less optimistic than 77%. Conditional on both signals, the expected value of the company to either bidder is $v(t_i, t_j)$. We assume $v(\cdot, \cdot)$ has strictly positive derivatives $\partial v / \partial t_k$ everywhere.

The company is sold using a conventional ascending bid (i.e. English) auction. That is, the price starts at zero and rises continuously. When one bidder drops out, the other bidder buys the fraction of the company that he does not already own at the current price per unit.[22] (If bidders quit simultaneously we assume the company is allocated randomly

[20] This assumption is consistent with takeover regulations that require bidders to disclose their stakes.

[21] Among the many factors that could affect the size of a bidder's toehold are the liquidity of the company's shares, institutional constraints such as the Williams Act and SEC rule 16(b) which may affect some bidders' ability to retain profits if a toehold of 10% or more is sold, the effect of accumulating shares on the likelihood of arranging a friendly deal (as in Freeman (1991)), the probability that management will find out that a toehold is being accumulated and the range of management response, the risk that information leakage about a potential offer will cause a pre-bid runup in the stock price (Schwert (1996) shows that a pre-bid runup forces a bidder to pay more to buy a company), and the amount of shares held by the bidder prior to any decision to make an offer for the company (many toeholders own large stakes accumulated years before a buyout offer).

[22] Thus all shareholders (including the two bidders) are assumed to be willing to sell out to the highest bidder so we are ignoring any free-rider problems of the kind discussed by Grossman and Hart (1980). Also, all offers are assumed to be binding (which is supported by the legal environments of the EC and US). Offers are for all the outstanding shares. (Partial offers are legal under dominant US law but only if they are non-discriminating and we would obtain similar results in this case.) See Burkart (1995) and

at the current price, though this assumption is unimportant.) Thus a (pure) strategy for bidder k is a price $b_k(t_k)$ at which he will quit if the other bidder has not yet done so.

We assume that $v(\cdot,\cdot)$ is symmetric in t_i and t_j. We define i's "marginal revenue" as $MR_i(t_i,t_j) \equiv v(t_i,t_j) - (1-t_i)\frac{\partial v}{\partial t_i}(t_i,t_j)$,[23] and assume that the bidder with the higher signal has the higher marginal revenue, i.e., $t_i > t_j \Longrightarrow MR_i(t_i,t_j) > MR_j(t_i,t_j)$. This is a standard assumption in auction theory and monopoly theory; it corresponds to assuming that bidders' marginal revenues are downward sloping in symmetric private-values auction problems and the corresponding monopoly problems. The assumption is a much stronger one for common-value auctions than for private-value auctions, [24] but we note that the assumptions of this paragraph are only required for Proposition 2 and 6.

We denote the price that the bidding has currently reached by b. We write bidder k's equilibrium profits, conditional on his signal, as $\pi_k(t_k)$, and his unconditional profits (averaged across his possible signals) as Π_k. We write the expected profits accruing to all the shareholders except the two bidders as Π_0.

3 Solving the Model

In this section, we first establish the necessary and sufficient conditions for the equilibrium strategies of our model (Lemmas 1 and 2), next solve for the equilibrium (Proposition 1), and then calculate the expected revenue of the bidders and the non-bidding shareholders.

By standard arguments, we obtain

Bidders' equilibrium strategies must be pure strategies $b_i(t_i)$ and $b_j(t_j)$ that are continuous and strictly increasing functions of their types with $b_i(0) = b_j(0) > v(0,0)$ and

McAfee, Vincent, Williams, and Havens (1993), p. 461, for more legal details.

[23]In analysing our auction using marginal revenues, we are following Bulow and Roberts (1989) who first showed how to interpret private-value auctions in terms of marginal revenues, and Bulow and Klemperer (1996) who extended their interpretation to common-values settings such as this one. Since the marginal revenue of a bidder is exactly the marginal revenue of the customer who is the same fraction of the way down the distribution of potential buyers in the monopoly model, this interpretation allows the direct translation of results from monopoly theory into auction theory and so facilitates the analysis of auctions and the development of intuition about them.

[24]See Bulow and Klemperer (1997) for discussion of when this assumption holds in the common-value case. See also Myerson (1981), who calls this the "regular" case in his largely private-value analysis, Bulow and Roberts (1989), who refer to this as downward-sloping marginal revenue in their private-value analysis, and Bulow and Klemperer (1996), who also (more loosely) refer to this as downward-sloping marginal revenue in the general case.

$b_i(1) = b_j(1) = v(1,1)$.[25]

We can therefore define "equilibrium correspondence" functions $\phi_i(\cdot)$ and $\phi_j(\cdot)$ by $b_i(\phi_i(t_j)) = b_j(t_j)$ and $b_j(\phi_j(t_i)) = b_i(t_i)$. That is, in equilibrium, type t_i of i and type $\phi_j(t_i)$ of j drop out at the same price, and type t_j of j and type $\phi_i(t_j)$ of i drop out at the same price. So bidder i will defeat an opponent of type t_j if and only if $t_j \leq \phi_j(t_i)$, and $\phi_j(t_i)$ is type t_i's probability of winning the company.

Given i's bidding function $b_i(\cdot)$, for any type t_j of j we can find t_j's equilibrium choice of where to quit or, equivalently, t_j's choice of which t_i to drop out at the same time as, by maximizing t_j's expected revenues

$$\max_{t_i} \left\{ \int_{t=0}^{t_i} \Big[v(t, t_j) - (1-\theta_j) b_i(t) \Big] \, dt + \theta_j (1 - t_i) b_i(t_i) \right\}. \tag{1}$$

The term in the integral is j's revenues from buying, and the second term is j's revenue from selling. Setting the derivative of (1) equal to zero[26] and using the fact that $t_j = \phi_j(t_i)$ in equilibrium yields

$$b_i'(t_i) = \frac{1}{\theta_j} \left(\frac{1}{1-t_i} \right) \Big[b_i(t_i) - v(t_i, \phi_j(t_i)) \Big]. \tag{2}$$

The logic is straightforward: given that the price has already reached $b_i(t_i)$, the benefit to j of dropping out against type $(t_i + dt_i)$ instead of type t_i is $\theta_j b_i'(t_i)\, dt_i$ — j's toehold times the increase in price per share earned by the later exit. The cost is that with probability $dt_i/(1-t_i)$, j will "win" an auction he would otherwise have lost, suffering a loss equal to the amount bid less the value of the asset conditional on both bidders being marginal.

[25] See Appendix for proof.

[26] Making this argument assumes $b_i(t_i)$ is differentiable. Strictly we should proceed by noting that type $t_j = \phi_j(t_i)$ prefers quitting at $b_i(t_i)$ to $b_i(t_i + \Delta t_i)$. Therefore

$$\theta_j \left[b_i(t_i + \Delta t_i) - b_i(t_i) \right] \left(1 - \frac{\Delta t_i}{1-t_i} \right) \leq \left(\frac{\Delta t_i}{1-t_i} \right) \big[b_i(t_i) - v\big(t_i, \phi_j(t_i)\big) \big] + o(\Delta b) + o(\Delta v),$$

in which $o(\Delta b)$ and $o(\Delta v)$ are terms of smaller orders than, respectively, $\Delta b \equiv b_i(t_i + \Delta t_i) - b_i(t_i)$ and $\Delta v \equiv \Delta t_i \cdot \frac{\partial v}{\partial t_i}$. So

$$\limsup_{\Delta t_i \to 0} \frac{b_i(t_i + \Delta t_i) - b_i(t_i)}{\Delta t_i} \leq \frac{1}{\theta_j} \left(\frac{1}{1-t_i} \right) \big[b_i(t_i) - v\big(t_i, \phi_j(t_i)\big) \big].$$

Using the fact that j's type $\phi_j(t_i + \Delta t_i)$ prefers quitting at $b_i(t_i + \Delta t_i)$ to $b_i(t_i)$ yields the same equation except with the inequality reversed and lim inf instead of lim sup, so the right derivative of $b_i(\cdot)$ exists and is given by (2). Examining the incentives for j's type $\phi_j(t_i)$ to quit at $b_i(t_i - \Delta t_i)$ and for j's type $\phi_j(t_i - \Delta t_i)$ to quit at $b_i(t_i)$ completes the argument by showing the left derivative exists and is also given by (2).

It is easy to check that (2) and the corresponding condition for $b_j'(t_j)$ are sufficient for equilibrium, i.e., satisfy global second-order conditions,[27] so we have:[28]

Necessary and sufficient conditions for the bidding strategies $b_i(t_i)$ and $b_j(t_j)$ to form a Nash equilibrium are that $b_i(\cdot)$ and $b_j(\cdot)$ are increasing functions that satisfy

$$b_i'(t_i) = \frac{1}{\theta_j}\frac{1}{1-t_i}\left[b_i(t_i) - v(t_i, \phi_j(t_i))\right], \tag{3}$$

$$b_j'(t_j) = \frac{1}{\theta_i}\frac{1}{1-t_j}\left[b_j(t_j) - v(\phi_i(t_j), t_j)\right], \tag{4}$$

where

$$\phi_i(\cdot) = b_i^{-1}\left(b_j(\cdot)\right) \quad \text{and} \quad \phi_j(\cdot) = b_j^{-1}\left(b_i(\cdot)\right),$$

with boundary conditions given by

$$b_i(0) = b_j(0) > v(0,0), \tag{5}$$

$$b_i(1) = b_j(1) = v(1,1). \tag{6}$$

Equation (3) can be integrated to yield

$$b_i(t_i) = \frac{1}{\theta_j}(1-t_i)^{-\frac{1}{\theta_j}}\left[k - \int_0^{t_i} v(t, \phi_j(t))(1-t)^{\frac{1}{\theta_j}-1}dt\right],$$

where k is a constant of integration. According to boundary condition (6), it is given by $k = \int_0^1 v(t, \phi_j(t))(1-t)^{\frac{1}{\theta_j}-1}dt$. So we have

$$b_i(t_i) = \frac{\int_{t_i}^1 v(t, \phi_j(t))(1-t)^{\frac{1}{\theta_j}-1}dt}{\int_{t_i}^1 (1-t)^{\frac{1}{\theta_j}-1}dt}. \tag{7}$$

Define $H_k(t_k)$ to be bidder k's hazard rate, that is, the instantaneous rate at which bidder k quits as the price rises divided by the probability k is still present. So $H_k(t_k) =$

[27] Assume, for contradiction, that at some bidding level type t_i's optimal strategy is to deviate to mimic type $t_i' > t_i$. Observe that at any point a higher type has a greater incentive than a lower type to remain in the bidding (the potential gains from selling out at a higher price are the same and the potential losses from being sold to are less). But the derivation of the first-order condition demonstrates that a type slightly below t_i' does not wish to stay in the bidding to mimic t_i' (see preceding note). So t_i prefers to mimic this type than to mimic t_i', which is a contradiction.

[28] Our working paper, Bulow, Huang, and Klemperer (1995), extends this lemma to a more general setting in which the bidders' valuations of the target company have both private- and common-value components, and proves existence and uniqueness of equilibrium for the general case.

$\frac{1/b_k'(t_k)}{1-t_k}$ since types are distributed uniformly. Since $b_i(t_i) = b_j(\phi_j(t_i))$, dividing equation (3) by equation (4) yields

$$\frac{H_i(t_i)}{H_j(\phi_j(t_i))} = \frac{\theta_j}{\theta_i}. \tag{8}$$

Since boundary conditions (5) and (6) imply that $\phi_j(0) = 0$ and $\phi_j(1) = 1$, the unique solution to (8) is

$$(1 - t_j)^{\theta_j} = (1 - t_i)^{\theta_i} \tag{9}$$

i.e.

$$\phi_j(t_i) = 1 - (1 - t_i)^{\theta_i/\theta_j} \tag{10}$$

Substituting into (7), we have:

Proposition 1. *There exists a unique Nash equilibrium. In it bidder i remains in the bidding until the price reaches*

$$b_i(t_i) = \frac{\int_{t_i}^{1} v(t, 1 - (1 - t)^{\frac{\theta_i}{\theta_j}})(1 - t)^{\frac{1-\theta_j}{\theta_j}}\, dt}{\int_{t_i}^{1} (1 - t)^{\frac{1-\theta_j}{\theta_j}}\, dt} \tag{11}$$

and bidder j's strategy can be expressed symmetrically.

Note that our equilibrium is unique, in stark contrast to the case without toeholds in which *every* different weakly increasing function $\phi_j(t_i)$ yields a distinct equilibrium, $b_i(t_i) = v(t_i, \phi_j(t_i)) = b_j(\phi_j(t_i))$ (see Milgrom (1981)). The reason is that the toeholds determine a precise relationship for each bidder between his opponent's hazard rate and the "markup" he will bid over what the company would be worth conditional on his opponent being of the lowest remaining type. Absent toeholds, these markups are zero and there is no restriction on the ratio of the hazard rates at any price.[29]

The easiest way to calculate bidder i's profits is to note, by the envelope theorem, that type $t_i + dt_i$'s profits can be computed to first order as if he followed type t_i's strategy, in

[29] More precisely, without toeholds, the two bidders' optimization conditions are degenerate and so cannot uniquely determine the two equilibrium strategies. Introducing toeholds breaks this degeneracy, giving two distinct optimization conditions which uniquely determine the equilibrium strategies.

which case he would earn t_i's profits, except that the company is worth $\frac{\partial v}{\partial t_i}(t_i, t_j)\, dt_i$ more when he wins against a bidder with signal t_j, so

$$\frac{d\pi_i(t_i)}{dt_i} = \int_{t_j=0}^{\phi_j(t_i)} \frac{\partial v}{\partial t_i}(t_i, t_j)\, dt_j$$

which implies

$$\begin{aligned} \pi_i(t_i) &= \pi_i(0) + \int_{t=0}^{t_i} \int_{t_j=0}^{\phi_j(t)} \frac{\partial v}{\partial t}(t, t_j)\, dt_j\, dt \\ &= \theta_i b_i(0) + \int_{t=0}^{t_i} \int_{t_j=0}^{\phi_j(t)} \frac{\partial v}{\partial t}(t, t_j)\, dt_j\, dt, \end{aligned} \tag{12}$$

since a bidder with $t_i = 0$ always sells at $b_i(0)$.

Obviously, bidder i's expected profits (after averaging across all possible values of his information and simplifying) are

$$\Pi_i = \int_{t_i=0}^{1} \pi_i(t_i) dt_i = \theta_i b_i(0) + \int_{t_i=0}^{1} \int_{t_j=0}^{\phi_j(t_i)} (1-t_i) \frac{\partial v}{\partial t_i}(t_i, t_j) dt_j dt_i. \tag{13}$$

The expected surplus accruing to all shareholders except the bidders is

$$\Pi_0 = \int_{t_i=0}^{1} \int_{t_j=0}^{1} v(t_i, t_j) dt_j dt_i - \Pi_i - \Pi_j, \tag{14}$$

and the average sale price is $\Pi_0/(1-\theta_i-\theta_j)$.

It is also useful to note that (13) can be written as

$$\Pi_i = \theta_i b_i(0) + \int_{t_i=0}^{1} \int_{t_j=0}^{1} p_i(t_i, t_j)(1-t_i) \frac{\partial v}{\partial t_i}(t_i, t_j) dt_j dt_i, \tag{15}$$

in which $p_i(t_i, t_j)$ is the probability with which i wins the company if the bidders' signals are t_i and t_j. So substituting $(p_i(t_i, t_j) + p_j(t_i, t_j))v(t_i, t_j)$ for $v(t_i, t_j)$, we can collect terms to rewrite (14) as

$$\begin{aligned} \Pi_0 &= \int_{t_i=0}^{1} \int_{t_j=0}^{1} \Big[\Big(v(t_i, t_j) - (1-t_i)\tfrac{\partial v}{\partial t_i}(t_i, t_j)\Big)\, p_i(t_i, t_j) \\ &\quad + \Big(v(t_i, t_j) - (1-t_j)\tfrac{\partial v}{\partial t_j}(t_i, t_j)\Big)\, p_j(t_i, t_j)\Big]\, dt_j dt_i - \theta_i b_i(0) - \theta_j b_j(0), \end{aligned}$$

or

$$\Pi_0 = E_{t_i,t_j}(MR_{\text{winning bidder}}) - \theta_i b_i(0) - \theta_j b_j(0), \tag{16}$$

in which MR_i is i's "marginal revenue" as defined in Section 2.

Linear Example

As an example we explicitly compute the case in which the company's value is just the sum of the bidders' signals, $v = t_i + t_j$. Performing the integration in (11) we have

$$b_i(t_i) = 2 - \frac{1}{1+\theta_j}(1-t_i) - \frac{1}{1+\theta_i}(1-t_i)^{\theta_i/\theta_j}. \tag{17}$$

Hence

$$\pi_i(t_i) = \theta_i\left(\frac{\theta_i}{\theta_i+1} + \frac{\theta_j}{\theta_j+1}\right) + t_i - \left(\frac{\theta_j}{\theta_i+\theta_j}\right)\left(1-(1-t_i)^{\frac{\theta_i+\theta_j}{\theta_j}}\right). \tag{18}$$

So also

$$\Pi_i = \theta_i\left(\frac{\theta_i}{\theta_i+1} + \frac{\theta_j}{\theta_j+1} + \frac{1}{2\theta_i+4\theta_j}\right), \tag{19}$$

$$\Pi_0 = 1 - (\theta_i+\theta_j)\left(\frac{\theta_i}{\theta_i+1} + \frac{\theta_j}{\theta_j+1}\right) - \left(\frac{\theta_i}{2\theta_i+4\theta_j}\right) - \left(\frac{\theta_j}{4\theta_i+2\theta_j}\right) \tag{20}$$

and the average sale price is

$$\left[\frac{\theta_j(2\theta_j+\theta_i+1)}{(\theta_j+1)(2\theta_j+\theta_i)}\right] + \left[\frac{\theta_i(2\theta_i+\theta_j+1)}{(\theta_i+1)(2\theta_i+\theta_j)}\right].$$

The bidding functions for this example are illustrated in Figure 1 for the case in which the toeholds are $\theta_1 = 0.05$ and $\theta_2 = 0.01$. Observe that the bidder with the larger toehold always bids more than in the symmetric equilibrium without toeholds, while the bidder with the smaller toehold bids less than if neither bidder had a toehold except for very low values of his signal. Figure 2 also shows the bidding functions when the toeholds are $\theta_1 = 0.10$ and $\theta_2 = 0.01$; increasing bidder 1's toehold makes that bidder bid more aggressively (and increases his expected profits) for all values of his signal.

<u>Figure 1 goes here.</u>

Figure 1: Equilibrium Bidding Functions With and Without Toeholds for Linear Example $v = t_1 + t_2$ with Toeholds of 5% and 1%.

<u>Figure 2 goes here.</u>

Figure 2: Equilibrium Bidding Functions With Different Size Toeholds
for Linear Example $v = t_1 + t_2$.
Dashed lines: bidding functions with toeholds of 10% and 1%;
Solid lines: bidding functions with toeholds of 5% and 1%.

The next section describes properties of the equilibrium, including those illustrated in the figures, that apply in the general case.

4 Properties of the Equilibrium

If there were no toeholds, type t_i would bid up to the price $v(t_i, \phi_j(t_i))$ at which he would just be indifferent about winning the auction, but it is immediate from equation (7) that every bidder except the highest possible type, $t_i = 1$, bids beyond this price.[30] So except for types $t_i = 1$ and $t_j = 1$, *any bidder who narrowly "wins" the auction loses money.*

From equation (8), bidder i always quits at a rate θ_j/θ_i times as fast as bidder j, so it follows immediately that i "wins" the auction, i.e. buys the company, with probability $\left(\frac{\theta_i}{\theta_i+\theta_j}\right)$. Thus *probabilities of winning the auction are highly sensitive to the relative sizes of bidders' stakes*, and a bidder's probability of winning is increasing in his stake.

It also follows that *increasing a bidder's stake increases his probability of winning, conditional on whatever information he has* (i.e. $\phi_j(t_i)$ is strictly increasing in θ_i for all $0 < t_i < 1$), and that if i's stake is smaller than j's, then bidder i will lose to any bidder j with equally optimistic, or not-too-much less optimistic, information than he has (i.e. $\theta_i < \theta_j \Rightarrow t_i > \phi_j(t_i)$ for all $0 < t_i < 1$).

Note, in particular, that *a bidder with zero stake has zero probability of winning.* To see why, observe that if at any point the lowest possible remaining types of i and j were known to be t_i and t_j, then i, with zero stake, will bid up to $v(t_i, t_j)$ while j, with positive stake, will bid strictly more. So whatever are the current lowest types, there are always more of i's types who must quit before any of j's types leave.

From (11), *increasing a bidder's stake always makes him bid more aggressively.* That is, $\dfrac{\partial b_i(t_i)}{\partial \theta_i} > 0$ for all $t_i < 1$. This is what we expect — a higher stake makes a bidder more

[30] Of course, this does not mean bidders necessarily bid more than if there were no toeholds, since the functions $\phi_k(\cdot)$ are different.

like a seller who wants to set a high price, than like a pure buyer who wants to buy low.

Since $b_i(0) = b_j(0)$ and bidding strategies are continuous, *all types of bidder j with sufficiently pessimistic information also bid more aggressively if i's stake is increased.* The intuition is that because i is bidding more aggressively, low types of bidder j should take the opportunity to bid the price up under him.

However, for higher types of bidder j it is not clear whether increasing i's stake should make j more or less aggressive: bidder j also has to take account of the larger winner's curse of winning against a more-aggressive bidder i. In fact, there is no general result about whether raising i's stake raises or lowers j's bid.[31]

Even though raising a bidder's stake makes some types of his opponent more aggressive — so results in lower ex-post profits for some types of the bidder — *increasing a bidder's stake always increases his expected profits, whatever his signal.* In fact, increasing a bidder's toehold increases his expected profits in two ways; it both raises the price $b_i(0)$ at which the bidding starts and at which the bidder can sell out if he has the lowest possible signal, and also increases the incremental surplus that he earns from any higher signal (since $\frac{d\phi_j}{d\theta_i}(t_i) > 0$ for all $0 < t_i < 1$).

Increasing i's toehold also increases j's profits if j's signal is below some critical level, since when j has a low signal he is likely to sell and if j sells he sells for a higher price. (In particular, $\Pi_j(0) = b_j(0)\theta_j = b_i(0)\theta_j$ is larger.) Conversely, however, *it reduces j's profits if his signal is above a certain level,* since when j buys he must pay more. (To check this, recall that $\frac{d\phi_i(t_j)}{d\theta_i} < 0$ and it is easy to see that $\frac{d\pi_j(1)}{d\theta_i} < 0$ since $t_j = 1$ always buys and always pays more if θ_i is larger.) *Overall, increasing i's toehold reduces the profits of j averaged over all j's types* (i.e. $\frac{d\Pi_j}{d\theta_i} < 0$).

The expected price conditional on winning is the same for both bidders (and equals the average sale price) - - - because the relative rates at which the two bidders quit is the same

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[31] It is easy to check for the linear case that

$$\frac{\partial b_j(t_j)}{\partial \theta_i} = \frac{(1-t_j)}{(1+\theta_i)^2}\left[1 + \frac{(1+\theta_i)^2}{(1+\theta_j)}\frac{\theta_j}{\theta_i^2}(1-t_j)^{\frac{\theta_j-\theta_i}{\theta_i}}\log(1-t_j)\right].$$

So an increase in the share of the bidder with the larger toehold leads to the opponent bidding more/less aggressively according to whether his type is below/above some cutoff level. An increase in the share of the bidder with the smaller toehold always results in both weak and strong types of the opponent bidding more aggressively while intermediate types bid less aggressively.

at every price.

Observe that when toeholds are small, $b_i(0) = b_j(0)$ is of (or less than the) order $\theta_i + \theta_j$, [32] so $(\theta_i b_i(0) + \theta_j b_j(0))$ is a term of second order in $\theta_i + \theta_j$. From (16), we therefore have $\Pi_0 \approx E_{t_i,t_j}(MR_{\text{winning bidder}})$. Furthermore, by our assumption that the bidder with the higher signal has the higher marginal revenue, the expected marginal revenue of the winner is maximized over all possible mechanisms if and only if the bidder with the higher signal always wins the auction, that is, only when toeholds are symmetric. So with sufficiently small toeholds the non-bidding shareholders' expected wealth is highest with equal toeholds; the more unequal the toeholds, the more likely it is that the bidder with the lower signal, hence lower marginal revenue, will win the auction, so the lower is the expected wealth of the non-bidding shareholders. [33]

More precisely,

Proposition 2. *The expected sale price is higher from an (ascending) auction when bidders' toeholds are in a more equal ratio than when bidders' toeholds are in a less equal ratio, if the toeholds are sufficiently small. (I.e., for any given $0 \leq \lambda_1 < \lambda_2 \leq 1$, there is a $\overline{\theta}$ such that the expected sale price with any $\theta_i < \overline{\theta}$ and $\theta_j = \lambda_2 \theta_i$ exceeds the expected sale price with θ_i and $\theta_j = \lambda_1 \theta_i$.)* [34]

With larger toeholds the terms $\theta_i b_i(0)$ and $\theta_j b_j(0)$ are non-trivial, so Π_0 is not just a function of the expected marginal revenue of the winner. However it remains true that for any given $\theta_i + \theta_j$, symmetric toeholds are most desirable from the viewpoint of the non-bidding shareholders provided the lowest possible bid, $b_i(0) = b_j(0)$, is not too much higher for asymmetric than for symmetric toeholds.

[32] For small θ_i and θ_j, we have $b_i(0) = b_j(0) \approx \frac{\partial v}{\partial t_j}(0,0)\theta_i + \frac{\partial v}{\partial t_i}(0,0)\theta_j$.

[33] More asymmetric toeholds may increase the expected wealth of the non-bidding shareholders if the bidder with the higher signal does not necessarily have the higher marginal revenue. An example is $v = t_i^3 + t_j^3$. The reason is that even a bidder with an arbitrarily tiny toehold has no reason to quit below the value the company would have if his opponent had the lowest possible signal. Getting this value from the bidder with the smaller toehold — who is equally likely to be the bidder with the better or the worse information — yields a higher price for this valuation function, than a more symmetric contest in which the price is more likely to be determined by the bidder with the worse information. For further discussion of the assumption that the bidder with the higher signal has the higher marginal revenue see Bulow and Klemperer (1997).

[34] See Appendix for proof.

Thus it seems likely that the expected sale price will typically be increasing as the relative sizes of the bidders' toeholds are made more equal, whatever are their absolute sizes, and this is confirmed in the linear example:

Example. *In the linear example, $v = t_i + t_j$, the expected price increases as the sizes of the toeholds are made more equal, for any fixed sum of the sizes.*[35]

Note that, if $\theta_i + \theta_j$ is small, Π_0 depends, except for terms of high order, only on the ratio $\theta_i : \theta_j$. (This ratio determines the correspondence functions $\phi_i(\cdot)$ and hence determines which bidder wins the company.) It follows that, while toeholds remain small, giving both bidders free shares that proportionately increase their stakes by diluting the remaining shareholders' holdings has no first-order effect on anyone's expected wealth (before they know their types); each bidder's gain from his additional stake is just cancelled by his loss from his opponent's more aggressive behavior. For example, in the linear example, giving away 5% of a company in equal shares to two bidders who previously had arbitrarily tiny equal toeholds costs the remaining shareholders less than 1/2% of their expected wealth.[36]

It also follows that *diluting the stock by giving free shares to the bidder with the smaller toehold can increase the expected sale price per share, that is, increase the non-bidding shareholders' wealth.*

In summary, even small toeholds can have a large effect on the competition between the bidders. A bidder with a large toehold bids more aggressively and wins the auction with a higher probability. If the bidders' toeholds are sufficiently asymmetric, the bidder with a smaller toehold can be forced to quit at a very low price and the auction can generate a much lower expected revenue for the non-bidding shareholders.

[35]Let $\theta_i = \psi x$ and $\theta_j = (1-\psi)x$. Then

$$\Pi_0 = \left(\frac{1+2z}{2+z}\right) - \left(\frac{1+2zx}{1+x+zx^2}\right)x^2$$

in which $z \equiv \psi(1-\psi)$. So Π_0 is increasing in z, and z is increasing in ψ for $0 \le \psi < \frac{1}{2}$ and decreasing in ψ for $\frac{1}{2} < \psi \le 1$. So, for fixed $\theta_i + \theta_j$, Π_0 always increases as toeholds become more symmetric. In particular, if $\psi = \frac{1}{2}$ (symmetric toeholds) $\Pi_0 \approx \left(\frac{2}{3} - x^2\right)$, while if $\psi = 0$ or $\psi = 1$ (only one bidder has a toehold) $\Pi_0 \approx \left(\frac{1}{2} - x^2\right)$.

[36]Of course, this result relies on $v(0,0) = 0$. More generally, giving away options with exercise price equal to the lowest possible value of the company, that is, $v(0,0)$, has the effects described.

5 Changing the Game: (A) First Price Auctions

Since the "winner's curse" effects we have described mean that the bidder with the larger toehold wins with a high probability and at a low price, it is natural to ask whether the alternative common auction format — the first-price sealed bid auction — performs any better from the viewpoint of the non-bidding shareholders.[37] (Of course, absent toeholds, first-price and ascending auctions yield the same expected revenue when buyers are symmetric (Myerson (1981), Riley and Samuelson (1981).)

In a first-price auction each bidder, $k = i, j$, independently makes a single "best and final offer", $\widetilde{b}_k(t_k)$ per unit, and the highest bidder buys the fraction of the company, $(1-\theta_k)$, that he does not already own at the share price he bid. In the equilibrium of this case, type t_j of j will choose to beat all of the opponent's types below t_i (by bidding $\widetilde{b}_i(t_i)$), where t_i is chosen to maximize j's expected revenues

$$\max_{t_i} \left\{ \int_{t=0}^{t_i} \left[v(t, t_j) - (1-\theta_j)\widetilde{b}_i(t_i) \right] \, dt + \theta_j \int_{t=t_i}^{1} \widetilde{b}_i(t) \, dt \right\}. \tag{21}$$

Setting the derivative equal to zero, and substituting $t_j = \widetilde{\phi}_j(t_i)$ (that is, letting $\widetilde{\phi}_j(\cdot)$ and $\widetilde{\phi}_i(\cdot)$ be the equilibrium correspondence functions) yields

$$\widetilde{b}_i'(t_i) = \frac{1}{1-\theta_j} \cdot \frac{1}{t_i} \left[v(t_i, \widetilde{\phi}_j(t_i)) - \widetilde{b}_i(t_i) \right]. \tag{22}$$

The intuition is that given that j decides not to beat types of i above t_i, bidding even lower to win against dt_i fewer types saves an additional $(1-\theta_j)\widetilde{b}_i'(t_i)\, dt_i$ in payments when j wins, but the cost is that with probability dt_i/t_i, j loses an auction he would otherwise have won and so foregoes $v(t_i, t_j) - \widetilde{b}_i(t_i)$.

Notice that this intuition, and so also the derivative (**??**), corresponds exactly to our original problem with the change of variable θ_k to $(1-\theta_k)$ and t_k to $(1-t_k)$ except in $v(\cdot,\cdot)$ for $k = i, j$. It follows that the arguments of Section 3 extend immediately to imply

Necessary and sufficient conditions for the bidding strategies $\widetilde{b}_i(t_i)$ and $\widetilde{b}_j(t_j)$ to form a Nash equilibrium for the first-price auction are that $\widetilde{b}_i(\cdot)$ and $\widetilde{b}_j(\cdot)$ are increasing functions with

$$\widetilde{b}_i'(t_i) = \frac{1}{1-\theta_j} \cdot \frac{1}{t_i} \left[v(t_i, \widetilde{\phi}_j(t_i)) - \widetilde{b}_i(t_i) \right], \tag{23}$$

[37] See note 8 for discussion of the practical feasibility of the first-price auction.

$$\widetilde{b}_j'(t_j) = \frac{1}{1-\theta_i} \cdot \frac{1}{t_j}\left[v(\widetilde{\phi}_i(t_j), t_j) - \widetilde{b}_j(t_j)\right], \tag{24}$$

where

$$\widetilde{\phi}_i(\cdot) = \widetilde{b}_i^{-1}\left(\widetilde{b}_j(\cdot)\right) \quad \text{and} \quad \widetilde{\phi}_j(\cdot) = \widetilde{b}_j^{-1}\left(\widetilde{b}_i(\cdot)\right),$$

with the boundary conditions given by

$$\begin{aligned} \widetilde{b}_i(0) &= \widetilde{b}_j(0) = v(0,0), \\ \widetilde{b}_i(1) &= \widetilde{b}_j(1) < v(1,1). \end{aligned} \tag{25}$$

Likewise, we have

$$\widetilde{\phi}_j(t_i) = t_i^{\frac{1-\theta_i}{1-\theta_j}}, \tag{26}$$

and

Proposition 3. *There exists a unique Nash equilibrium of the first-price auction. In it i bids*

$$\widetilde{b}_i(t_i) = \frac{\int_0^{t_i} v(t, t^{\left(\frac{1-\theta_i}{1-\theta_j}\right)}) t^{\frac{\theta_j}{1-\theta_j}}\, dt}{\int_0^{t_i} t^{\frac{\theta_j}{1-\theta_j}}\, dt} \tag{27}$$

and bidder j's bid can be expressed symmetrically.

Now (26) implies that i "wins" with probability $\left(\dfrac{(1-\theta_j)}{(1-\theta_j)+(1-\theta_i)}\right)$. If $\theta_i > \theta_j$, this is smaller than the probability $\left(\dfrac{\theta_i}{\theta_i + \theta j}\right)$ with which i would win the ascending auction, so it also follows that

Proposition 4. *The probability that the bidder with the higher signal wins the auction is greater in the first-price auction than in the ascending auction.*

Thus the outcomes of first-price auctions are less sensitive to toeholds than are the outcomes of ascending auctions, although it remains true that the bidder with the larger toehold has a higher probability of winning.

The intuition is that a bidder with a toehold still has an incentive to bid higher than otherwise: bidding more aggressively is less costly when winning the auction means buying only fraction $(1-\theta)$ rather than all of the company. However, this effect is generally small unless θ is close to 1 (in which case the bidder has control anyway[38]). Furthermore, and more importantly, the indirect or "strategic" effect due to the winner's curse on the opponent is much smaller in first-price than in ascending auctions.[39] So the extreme outcome of the ascending auction, that a bidder with a relatively small toehold is almost completely driven out of the bidding, does not arise in the first-price auction.

Because toeholds provide greater incentives for bidding aggressively in ascending auctions than in first-price auctions, ascending auctions yield higher prices on average when toeholds are symmetric:

Proposition 5. *With symmetric toeholds, the expected sale price is higher in an ascending auction than in a first-price auction.* [40]

However, when toeholds are very asymmetric, the winner's curse effect that the bidder with the smaller toehold is forced to quit at a very low value in an ascending auction, implies first-price auctions are likely to perform better.

Proposition 6. *With asymmetric toeholds, the expected sale price is higher in a first-price auction than in an ascending auction, if the toeholds are sufficiently small. (I.e. for any* $\lambda \neq 1$, *the first-price auction yields a higher expected price for all* θ_i, θ_j *such that* $\theta_j = \lambda\theta_i \leq \overline{\theta}$, *for some* $\overline{\theta}$.)[41]

A more formal way to understand Propositions 5 and 6 is to recall that the expected sale price equals $\Pi_0/(1-\theta_i-\theta_j)$ and Π_0 can be written as in (16) for the ascending

[38] Our model therefore assumes $\theta < \frac{1}{2}$.

[39] In an ascending auction, when bidder i bids more aggressively, bidder j must bid less, because conditional on winning at any price his revenue is lower. (That is, bidding strategies are "strategic substitutes" in the terminology introduced by Bulow, Geanakoplos, and Klemperer (1985).) In a first-price auction, by contrast, bidder j's response to bidder i bidding more is ambiguous: when i bids more, j wants to bid *less* on the grounds that his marginal profit when he wins is lower, but *more* on the grounds that his probability of winning is lower so increasing his bid is less costly. So the ascending-auction logic that when i bids a little more, j bids a similar amount less, so i bids a similar amount more, so j bids a similar amount less, etc., does not apply in first-price auctions.

[40] See Appendix for proof. This result does not depend on the assumption of pure common values. Singh (1995) obtains this result for the pure private-values case.

[41] See Appendix for proof.

auction. By a exactly similar logic, Π_0 for the first-price auction can also be written as in (16) except that the term $b_i(0)$ is replaced by the expected price received by bidder i in a first-price auction if i has the lowest possible signal, that is $\int_{t_j=0}^{1} \widetilde{b}_j(t_j)dt_j$, and the term $b_j(0)$ is replaced similarly. There are therefore two differences between a first-price auction and an ascending auction:

First, the price received by bidder i with signal zero in a first-price auction $\left(\int_{t_j=0}^{1} \widetilde{b}_j(t_j)dt_j\right)$ is the average bid of a bidder j who does not know i's signal, whereas in an ascending auction bidder i must drop out immediately at $b_i(0)$. When toeholds are symmetric this is the only distinction between the expressions for Π_0 for the two types of auction, so the ascending auction yields higher prices for symmetric toeholds (Proposition 5).

Second, as Proposition 4 demonstrates, the first-price auction is won by the bidder with the higher signal in more cases than in the ascending auction, so the first-price auction is more often won by the bidder with the higher marginal revenue and so is likely to have the higher expected marginal revenue of the winning bidder.[42] In the limit as toeholds became arbitrarily tiny, this is the only distinction between the expressions for Π_0 for the two types of auction, so we expect the first-price auction to yield higher prices for asymmetric toeholds if the toeholds are not too large (Proposition 6).[43]

If bidders' toeholds are neither small nor symmetric, the sale-price comparison between the two auction forms is ambiguous, but our leading example suggests that first-price auctions are likely to be better in practice if there is much asymmetry in the relative sizes of the toeholds.

Example. *In the linear example $v = t_i + t_j$, a sufficient condition for the expected price to be higher in a first-price auction than an ascending auction is $\theta_i < \frac{1}{8}\,\theta_j$ or $\theta_i > 8\,\theta_j$. If $\theta_k < 0.1, k = i, j$, a sufficient condition is $\theta_i < \frac{1}{4}\,\theta_j$ or $\theta_i > 4\,\theta_j$.*

[42] However this need not be the case, even under our assumption that the bidder with the higher signal has the higher marginal revenue, because it is *not* true that the higher signal wins in the first-price auction in every case in which it wins in the ascending auction.

[43] An example which shows that if the bidder with the higher signal does not always have the higher marginal revenue, then an ascending auction may always yield a higher expected price than a first-price auction is $v = t_i^3 + t_j^3$.

6 Changing the Game: (B) Selling a Second Toehold

An alternative approach to compensating for the advantage that a bidder with a toehold has is to "level the playing field" by selling shares (or equivalently options) to the second bidder so that he has an equal stake.[44] Even if these shares are sold very cheap (so that all types of the second bidder will wish to buy them) the likely higher price from a fairer contest may more than outweigh the cost to the remaining shareholders of diluting their stake.

For example, with the linear value function $v = t_i + t_j$, if just one of the two bidders has a toehold, say θ, the expected profits of the non-bidding shareholders are $\frac{1}{2} - \frac{\theta^2}{1+\theta}$ (from (20)). The bidder without the toehold makes zero expected profit (whatever his signal) so, even if he had the lowest possible signal, he would be prepared to pay $\frac{2\theta^2}{1+\theta}$, that is, $\theta b(0)$ when both bidders have a stake of θ, for a stake of equal size. The expected profits of the non-bidding shareholders would then be $\frac{2\theta^2}{1+\theta}$ plus the expected profits from the bidding, $\frac{2}{3} - \frac{4\theta^2}{1+\theta}$, which equals $\frac{2}{3} - \frac{2\theta^2}{1+\theta}$ in all. This exceeds the expected profits if there were no such sale, $(\frac{1}{2} - \frac{\theta^2}{1+\theta})$, for all $\theta \leq \frac{1}{2}$.

In fact, even if the stake could only be given away free,[45] giving away the stake would dominate not doing so for all $\theta \leq \frac{1}{4}$.[46] [47]

7 Asymmetric Value Functions

Our analysis thus far has assumed that the value function is symmetric in bidders' signals, that is, that the bidders have equally valuable private information about the value of the company. In fact, none of our analysis depends on this assumption. However, if the value function is not symmetric, it is implausible that the bidder with the higher signal will always have the higher marginal revenue, and dropping this assumption requires dropping

[44] Selling shares at price p is equivalent in this context to giving options for the same number of shares at exercise price p.

[45] Note that we have set the base price of the stock to be zero if both bidders observe the lowest possible signal. So "given away free" here means selling them at the base price of the stock.

[46] Thus selling shares, or giving options, at a price close to the lowest possible value of the company may be acceptable management behavior in a context in which the value function is hard to assess.

[47] In fact selling, or giving, a second toehold is even more desirable than this if it is done through e.g. issuing new shares that dilute the size of the first bidder's stake, rather than by just selling a fraction of the non-bidding shareholders' shares. Dilution is probably more realistic but it was not needed for our result.

Propositions 2 and 6. (Propositions 1, 3, 4, and 5 are unaffected; they depend neither on the value function being symmetric, nor on any assumption about marginal revenues.)

If the bidders' information is not equally valuable, then the bidder to whose information the value is less sensitive (the bidder, k, with the lower $\frac{\partial v}{\partial t_k}(\cdot,\cdot)$) will typically have a higher marginal revenue when $t_i = t_j$, that is, when each bidder receives a signal that is the same fraction of the way down the distribution of signals that he could have received. Therefore, by contrast with Proposition 2, an auction in which the low-information bidder has the larger toehold and so sometimes wins when he has the lower signal may yield higher expected revenue than an auction with symmetric toeholds.[48] Similarly, by contrast with Proposition 6, if the low-information bidder has the larger toehold, an ascending auction may be preferred to a first-price auction, since the ascending auction gives a greater bias in favour of the larger toeholder's probability of winning. Of course, if the low-information bidder also has the smaller toehold, then an ascending auction will be particularly disastrous.

8 Conclusion

Toeholds can dramatically influence takeover battles. A bidder with a large toehold will have an incentive to bid aggressively, essentially because every price she quotes is both a bid for the rest of the company and an ask for her own shares. This increased aggressiveness will cause a competitor to alter his strategy as well. A competitor with a smaller toehold who is relatively pessimistic about the value of the company will become more aggressive, counting on the large toeholder to buy him out at a higher price. If the competitor has an optimistic assessment of the company's prospects, though, the large toeholder's aggressive strategy will cause the competitor to become more conservative, because of an exacerbated winner's curse.

Because toeholds make a bidder more aggressive, which can make a competitor more conservative, which can make the bidder still more aggressive, and so on, even small toeholds can have large effects. A toehold can sharply improve a bidder's chance of winning an

[48] For example, if the value function is linear but twice as sensitive to i's signal as to j's signal (i.e., $v = 2t_i + t_j$), then in the limit as all toeholds become tiny, the expected sale price is maximized when j's toehold is approximately three times as large as i's toehold.

auction, and raise the bidder's expected profits at the expense of both other bidders and stockholders.

The strategic consequences that so benefit the toeholder create a problem for a board of directors interested in attaining the highest possible sales price for their investors. The board of a target company may therefore wish to "level the playing field" by selling a toehold to a new bidder, or by changing the rules of the auction.

Appendix: Proofs

Proof of Lemma 1: Let $\bar{B}$ be the lowest price level at or below which, with probability 1, at least one bidder has dropped out. It is easy to see that if a low type gets the same expected surplus from two different quitting prices and the lower price is below $\bar{B}$, then a higher type always strictly prefers the higher quitting price. So at least up to $\bar{B}$, higher types quit (weakly) after lower types.

Define the common bidding range as price levels below $\bar{B}$.

Now if i has an "atom" (that is, an interval of his types drops out at a single price) within the common range, then j cannot have an atom at the same price, since an interval of j's types cannot all prefer to quit simultaneously with i's atom rather than leave either just before or just after.

We next argue that the equilibrium bidding functions $b_i(t_i)$ and $b_j(t_j)$ are single-valued and continuous on the common range, that is, there are no "gaps" (no intervals of prices within the common range within which a bidder drops out with probability zero). The reason is that if i has a gap, then j would do better to raise the price to the top of the gap (thus raising the price j receives for his share) than to drop out during the gap. So j must have a gap that starts no higher than the start of i's gap. Furthermore, unless i has an atom at the start of the gap, j would do better to raise the price to the top of the gap than to drop out just below the start of i's gap, that is, j's gap starts lower than i's. So, since we have already shown that i and j cannot both have atoms at the same price, we obtain a contradiction.[49]

Similarly it follows that $b_i(0) = b_j(0)$, since if $b_i(0) > b_j(0)$, then type 0 of bidder j would do strictly better to increase his bid a little.

Now, observe that if i has an atom in the common range, there cannot be a t_j that is willing to drop out just after the atom quits; t_j would either prefer to quit just before the atom (if t_j's value conditional on i being among the types within the atom is less than the current price) or prefer to quit a finite distance later (since t_j's lowest possible value conditional on i being above the atom must otherwise strictly exceed the current price). So, since we have already shown there are no gaps, any atom must be at the top of the

[49] Note that without toeholds, gaps would be feasible, since a bidder who knows he will be the next to drop out is indifferent about the price at which he does so.

common bidding range.

It now follows that $b_i(0) = b_j(0) > v(0,0)$, since if not then type 0 of bidder j would do better to raise his bid slightly; raising his bid by ε gains $\varepsilon\theta_j$ when he still sells (with probability close to 1) and loses less than $\varepsilon(1-\theta_j)$ when he ends up buying (which happens with a probability that can be made arbitrarily small by reducing ε).

At the top of the common range, assume, without loss of generality, that j is the player who quits with probability 1 by or at price $\bar{B}$. Then, for some $\hat{t}_i$, the types (and only the types) $t_i \geq \hat{t}_i$ of i quit at or above $\bar{B}$ (by the argument in the first paragraph of this proof). Then $\bar{B} \geq v(\hat{t}_i, 1)$ (so that it is always rational for j to sell at $\bar{B}$). But also $\bar{B} \leq v(\hat{t}_i, 1)$ (because either type $t_i = \hat{t}_i$ is willing to buy at $\bar{B}$ with probability 1; or if type $\hat{t}_i$ is not buying with probability 1, then j must have an atom at $\bar{B}$ and $\hat{t}_i$ is bidding $\bar{B}$, so $\bar{B} \leq v(\hat{t}_i, 1)$ otherwise $\hat{t}_i$ will quit just before j's atom). So $\bar{B} = v(\hat{t}_i, 1)$. Now we can't have $\hat{t}_i < 1$ or j's types just below 1 would prefer quitting just after $\bar{B}$ to just before $\bar{B}$; either i has an atom at $\bar{B}$ so buying just above $\bar{B} = v(\hat{t}_i, 1)$ is profitable, or i does not have an atom so raising t_j's bid by ε gains $\varepsilon\theta_j$ when he still sells (with probability close to 1, conditional on having reached price $\bar{B} = v(\hat{t}_i, 1)$) and loses less than $\varepsilon(1-\theta_j)$ when he ends up buying (which happens with a probability that can be made arbitrarily small by reducing ε). So $\bar{B} = v(1,1)$, and it is straightforward that neither player can have an atom at this price (no type below 1 would wish to win with probability 1 at this price).[50]

Finally, since we showed that there can be no interval within the bidding range within which a bidder quits with probability zero, note that bidders cannot choose mixed strategies. $\square$.

Proof of Proposition 2: Since the correspondence function $\phi_j(t_i)$ is independent of θ_i for any given ratio $\theta_i : \theta_j, E_{t_i,t_j}(MR_{\text{winning bidder}})$ is also independent of θ_i for any given ratio and is strictly lower for the ratio λ_1 than the ratio λ_2 by our assumption that $t_i > t_j \Longrightarrow MR_i > MR_j$. But for any λ_1 or λ_2, $\lim_{\theta_k \to 0} \pi_k(0) = 0, k = i, j$, so the result follows straightforwardly from (16). $\square$.

[50]Note that we have only shown that players quit by $\bar{B}$ with probability 1. Strictly speaking, in a Nash equilibrium, the (zero-probability) types $t_i = 1$ and $t_j = 1$ can quit above $\bar{B}$, since it is a zero-probability event that the price will reach $\bar{B}$. (In a perfect Bayesian equilibrium, however, all types including $t_i = 1$ and $t_j = 1$ must quit by $\bar{B}$.)

Proof of Proposition 5: Using the argument leading up to (14), the expected sale price in the second-price auction is

$$\frac{1}{(1-\theta_i-\theta_j)}\left\{\int_{t_i=0}^{1}\int_{t_j=0}^{1} v(t_i,t_j)dt_j dt_i - \left[\pi_i(0)+\int_{t_i=0}^{1}\int_{t_j=0}^{\phi_j(t_i)}(1-t_i)\frac{dv}{dt_i}(t_i,t_j)dt_j dt_i\right]\right.$$
$$\left.-\left[\pi_j(0)+\int_{t_j=0}^{1}\int_{t_i=0}^{\phi_i(t_j)}(1-t_j)\frac{dv}{dt_j}(t_i,t_j)dt_i dt_j\right]\right\}$$

By the same logic, the expected sale price in the first-price auction is the same expression but substituting $\widetilde{\phi}_k(\cdot)$ for $\phi_k(\cdot)$ and $\widetilde{\pi}_k(0)$ for $\pi_k(0), k=i,j$, in which $\widetilde{\pi}_k(0)$ is bidder k's surplus when k has his lowest possible signal. If $\theta_i=\theta_j=\theta$ then $\phi_j(t_i)=\widetilde{\phi}_j(t_i)=t_i$, so the difference between these expressions is

$$\frac{1}{(1-2\theta)}\left\{\widetilde{\pi}_i(0)+\widetilde{\pi}_j(0)-\pi_i(0)-\pi_j(0)\right\}.$$

Substituting $\pi_i(0)=\theta_i b_i(0)$ and $\widetilde{\pi}_i(0)=\theta_i\int_{t_j=0}^{1}\widetilde{b}_j(t_j)dt_j$ (since a bidder with signal zero always sells) yields (after evaluating $\int_{t_j=0}^{1}\widetilde{b}_j(t_j)dt_j$ by parts) that this difference is

$$\frac{1}{(1-2\theta)}\int_{t=0}^{1}2v(t,t)\left\{\left[(1-t)-(1-t)^{\frac{1-\theta}{\theta}}\right]-\left[t^{\frac{\theta}{1-\theta}}-t\right]\right\}dt.$$

This is positive since $v(t,t)$ is monotonic increasing in t and the expression in curly brackets has expected value zero and is negative for all $t\in(0,\widehat{t})$ and positive for all $t\in(\widehat{t},1)$, for some $\widehat{t}$. $\square$.

Proof of Proposition 6: For a given λ, write $E(\lambda)$ and $\widetilde{E}(\lambda)$ for the values of $E_{t_i,t_j}(MR_{\text{winning bidder}})$ for the ascending auction and first-price auction, respectively. $E(\lambda)$ is independent of θ_i (since $\phi_i(\cdot)$ is independent of θ_i), while $\widetilde{E}(\lambda)$ is monotonic continuous decreasing in θ_i with $\lim_{\theta_i\to 0}\widetilde{E}(\lambda)=E(1)$, since $\widetilde{\phi}_j(t_i)=t_i^{\frac{1-\theta_i}{1-\lambda\theta_i}}$ is monotonic and continuous in θ_i for every t_i and $\lim_{\theta_i\to 0}\widetilde{\phi}_j(t_i)=t_i$ for every t_i. Furthermore, by our assumption that $t_i>t_j\Longrightarrow MR_i>MR_j$, $E(1)>E(\lambda)$ for all $\lambda\neq 1$. Finally it is straightforward that $\lim_{\theta_k\to 0}\pi_k(0)=\lim_{\theta_k\to 0}\widetilde{\pi}_k(0)=0, k=i,j$, for all λ, so the result follows easily from (16). $\square$.

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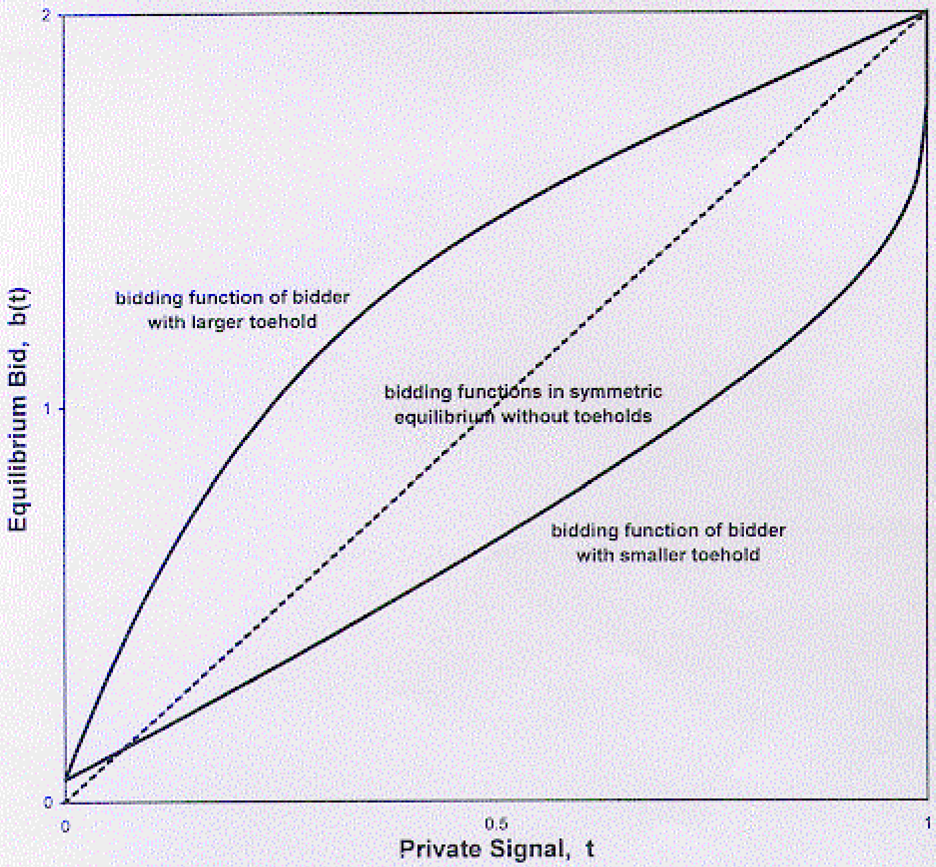

Figure 1: Equilibrium Bidding Functions With and Without Toeholds for Linear Example $v = t_1 + t_2$.

- - - - - : bidding functions in symmetric equilibrium without toeholds;

——— : bidding functions with toeholds of 5% and 1%.

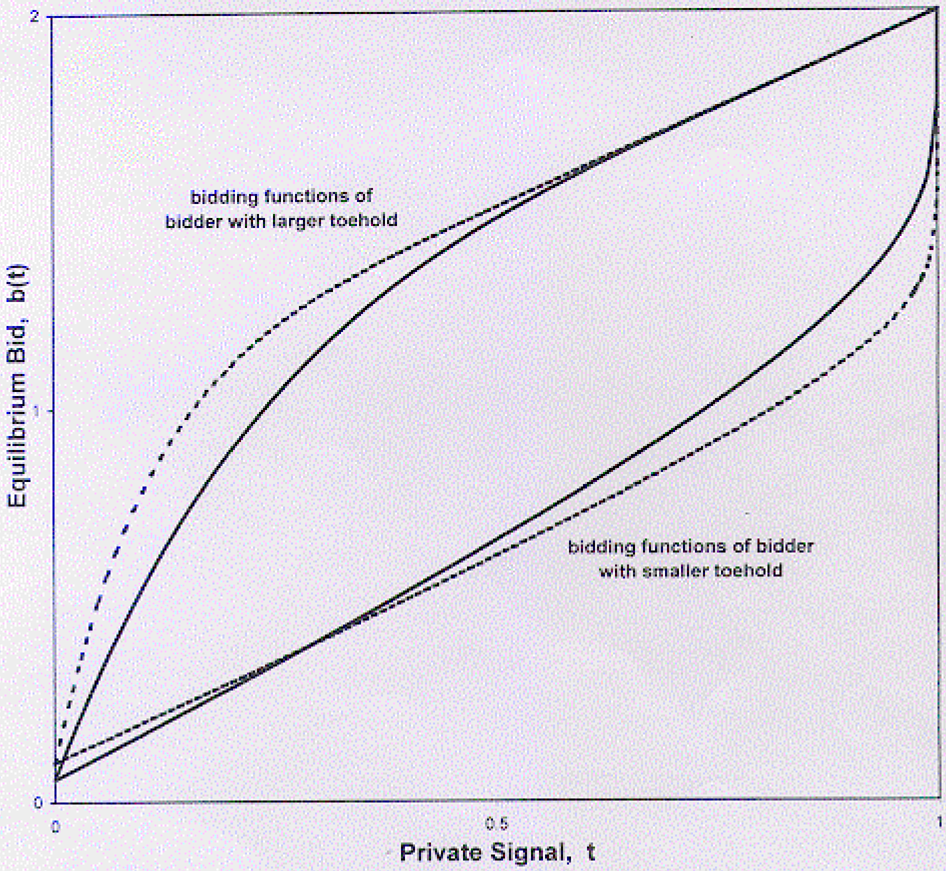

Figure 2: Equilibrium Bidding Functions With Different Size Toeholds for Linear Example $v = t_1 + t_2$.

- - - - - : bidding functions with toeholds of 10% and 1%;

——— : bidding functions with toeholds of 5% and 1%.