Wage-Tenure Contracts in a Frictional Labour Market:

Firms' Strategies for Recruitment and Retention

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Abstract

This paper analyses the contract-posting equilibrium in a general equilibrium matching model of the labour market with on-the-job turnover. Privately optimal contracts have a rising wage-tenure profile, even when productivity is constant. The effect is to reduce equilibrium turnover; when jobs differ in productivity, turnover is below the level required for efficient matching of workers to jobs. Contracts with a rising wage-tenure profile can be interpreted as a form of monopsonistic price discrimination. Credit constraints, or institutions that limit firms' ability to price discriminate, can improve the efficiency of matching.

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1. Introduction

The objective of this paper is to investigate the choice of employment contract by firms in a frictional labour market, and to demonstrate the implications of their choices for labour turnover and the efficiency of the market in allocating labour resources. When there are matching frictions, the time costs for workers and firms of finding alternative partners endow each existing relationship with a match rent. The typical assumption in general equilibrium search and matching models is that the sharing of the rent is determined through bilateral bargaining under conditions of symmetric information (as in, for example, Pissarides, 1990). Alternatively, in the wage-posting model (Diamond, 1971; Burdett and Mortensen, 1998), firms are assumed to possess all the bargaining power and "post" the wage that they will pay; individual workers observe the wage when they meet the firm, and decide whether to work there.

The main application of these models has been to the analysis of unemployment and macroeconomic policy (Mortensen and Pissarides, 1999). However, search and matching models provide a natural environment for investigating microeconomic issues of wage and contract determination. One of the primary functions of an employment contract is to regulate turnover when asymmetries of information make bilateral bargaining infeasible. In a standard two-period partial equilibrium model (Hall and Lazear, 1981) a single worker and firm choose a contract in the first period to maximise their joint expected surplus, with uncertain, but exogenously determined, alternative opportunities in the second period. The outcome is by definition constrained-efficient. But in a dynamic general equilibrium matching model we can investigate how the decisions of individual firms and workers affect the operation of the labour market as a whole. The choice of contract in one match affects the probability that the match will break up, and hence the likelihood that other matches will form. From the perspective of the firm, the optimal strategy for recruitment depends on other firms' strategies for retention, and vice-versa. The interaction of these strategies determines labour turnover, and the allocation of labour resources.

In this paper we consider a labour market with matching frictions, in which the productivity of a match is known with certainty. Firms can post general contracts, which workers accept or reject. Workers can move from one firm to another if offered a better contract. In this environment it is not optimal simply to post a wage, even if match productivity is constant. Optimal contracts for the firm have low value to the worker initially, and higher value subsequently: a rising wage-tenure profile. Although there may be many optimal contracts, we show that they are all equivalent in their implications for equilibrium turnover to a simple "step-contract", in which the wage is initially at the lowest level acceptable to the worker, and then steps up to the level of productivity. This result is used to determine equilibrium turnover for two particular cases. First, when all jobs are identical, there is no turnover in equilibrium. (This is in contrast to the result of Burdett and Mortensen (1998) who show that in the wage-posting equilibrium there is wage dispersion and turnover). In this case, wage-tenure contracts may enhance both private and social efficiency by avoiding wasteful turnover costs. In the second case, jobs differ in productivity. Then, the use by firms of a rising wage-tenure profile, while privately optimal, inhibits efficient matching: turnover is below the socially optimal level.

The adoption by firms of policies linking wages and tenure is a central element of the descriptive theory of internal labour markets (ILMs) (Doeringer and Piore, 1971), characterised by low mobility, with limited, low-level, ports of entry to firms, and well-defined career paths within firms. There are few formal models, but the primary rationale for ILMs is usually supposed to be specific investment (Wachter and Wright,1990). Interpretations differ (Ulman,1992): one view is that “inflexible” wage structures emerge in firms insulated from market forces, that these are reinforced by “convention”, and should be regarded as impediments to efficient employment. Wachter and Wright, in contrast, emphasise that the contractual arrangements in ILMs can be interpreted as efficient responses to specificity and information problems: “the ILM exists because it furthers the utility and profit maximising goals of the parties”. The model developed here contributes to this debate, in that the equilibrium contracts can be interpreted as ILM strategies. They arise from conditions of less than perfect competition; they are privately optimal; and they have both costs and benefits for labour market efficiency.

1.1 Related Theory and Evidence

Labour contracts specifying a low wage for an initial period, then subsequently the standard wage for the job, are a common device, particularly for unskilled jobs. Manning (2000), in a study of recruitment to low-wage jobs by five British firms, found that recruits could expect (on average) to wait 21 weeks before receiving a first pay-rise of 6%.

Andrews, Bradley and Upward (1999), studying the youth labour market in Lancashire, were able to record “advertised wage growth” for 29% of their sample of jobs. Amongst jobs that did not provide any training, average advertised wage growth was 17% during the first year. For higher level jobs, large firms often use an incremental pay scale, with new employees entering at a low point, and receiving wage increments at regular predetermined intervals until they reach the top of the scale.

There are two established theoretical explanations for such contracts: that they reflect the acquisition of specific human capital (Hashimoto, 1981), or delayed compensation as the solution to a shirking problem (Lazear, 1981). The interpretation offered in this paper has more in common with the specific capital model (since match rents can be regarded as specific capital), but differs in two important ways. First, rising wage-tenure profiles are used even when productivity remains constant throughout the duration of a match. Second, they may be detrimental to market efficiency.

In many labour market models, including shirking models, the first-best solution to the contracting problem is a lump-sum transfer from worker to firm, followed by a constant wage. Such arrangements are rarely observed, and it is sometimes argued that rising wage-tenure profiles are a second-best alternative if, for example, employees are credit-constrained. This hypothesis has been very little explored, and proponents of efficiency wage models have typically claimed that slowly rising wages cannot solve the efficiency wage problem[2]. In this paper we identify a second-best contract precisely, and examine its implications.

The model developed here is based on the random matching model of Burdett and Mortensen (1998)[3], which represents a significant innovation in the search and matching literature, in that employed workers may receive alternative job offers. We differ from Burdett and Mortensen in allowing firms to post general contracts rather than constant wages, and obtain different results. Another paper that allows firms to use general contracts in a general equilibrium matching model is Shimer (1996). His focus is on equilibrium unemployment and the efficient response to negative productivity shocks, and there is no job-to-job turnover. Consequently equilibrium contracts are very different from

[2] For example, Salop (1979) dismissed the possibility, in the context of a recruitment and retention model that has some features in common with the model in this paper.
[3] The model is also used by Mortensen and Vishwanath (1991), and Manning (1993).

those in this paper[4]; also, since there are no information asymmetries between the contracting parties, the equilibrium achieves full private and social efficiency.

2. The Model

There is a mass of identical workers and a mass of firms each of measure one. Over time, workers and firms meet each other at Poisson rate λ, which is independent of whether they are currently matched. A firm can match with several workers simultaneously, but a worker must terminate any existing match in order to enter a new one. Workers exit from the labour market (retire) at Poisson rate μ, again independent of employment status, and are replaced by new workers who are initially unemployed.

An unemployed worker has flow utility equal to b, which is constant across workers. The firm incurs a turnover cost $s \geq 0$ when establishing a new match; this can be interpreted as an initial specific training cost. Thereafter, the worker has flow productivity $p>b$, remaining constant over time. p and s may vary between firms; the distribution of (p,s) has bounded support[5].

On meeting a worker, the firm makes a take-it-or-leave-it offer of a contract. We assume that contracts must permit either party to terminate the relationship at any time. Workers and firms maximise the expected present value of income, and a worker moves from one match to another if and only if he is made strictly better off. We will make the convenient (but not substantive) assumption that an unemployed worker will accept an offer when he is indifferent between the contract and unemployment.

2.1 Symmetric Information and Efficiency

Suppose that, when a firm and worker meet, and throughout the duration of a match, each party knows everything relevant about the other. On meeting, they both observe the flow productivity of the match, p, and the firm observes whether the worker is currently matched, and, if so, his current productivity and the details of his current contract. Subsequently, they both observe alternative potential matches encountered by the worker.

[4] Broadly, they are "downward-sloping", in that they have highest value to the worker at low tenure.
[5] All of the general results in section 3 extend easily to a more general productivity profile $p(T)$, where T is the duration of the match. However, by focusing on a profile with initial low value, due to the cost s, and subsequently constant productivity, we simplify the analysis while capturing most of the insight to be gained from the more general case.

Under these conditions there is a multiplicity of strategies available to firms who can make take-it-or-leave-it contract offers, all of which lead to the same employment decisions in equilibrium, and all of which enable the firms to extract the rents: they are perfectly-price-discriminating monopsonists. With "offer-matching", for example, the firm initially offers a wage just sufficient to attract the worker (if it is able to do so) and then matches any subsequent offers, again if possible, with a wage sufficient to retain him. Alternatively, the firm could offer a wage equal to productivity, and then set an entry fee just small enough to induce the worker to accept. These, and many alternative strategies, support equilibria that are socially efficient: that is, workers change jobs if and only if they become more productive by doing so; hence total output is maximised, subject to the constraints imposed by the matching frictions.[6]

2.2 Information Assumptions and Game Structure

In the following analysis we suppose instead that, when a worker meets a firm, the firm does not observe the worker's existing contract, if any, or whether he is employed. Furthermore, the worker's current employer does not observe the offer made by an alternative employer, or even that a meeting is taking place.

The effect of these assumptions is that the firm cannot distinguish between workers at the time of meeting so must offer the same contract to all workers. Since it does not observe subsequent meetings, nothing in the contract can be contingent on such meetings occurring. In fact the only verifiable quantities and events upon which the arrangements in the contract *can* be contingent are the worker's tenure in the job, and the termination of the match by either party.

Given the dynamic nature of the model the firm may want to change its contract offer as the distribution of workers over jobs changes. However, like Burdett and Mortensen, we will focus only on the steady-state. The model is analysed as a one-shot game, in which each firm chooses its contract-offer strategy at the beginning of time. There may be an initial period in which the distribution of workers over jobs changes, until it reaches a steady-state, but firms do not discount the future, so they choose strategies that maximise their steady-state profits.

[6] In fact, under symmetric information, the use of a binding contract is not necessary for efficiency. Offer-matching is equivalent to continuous bilateral bargaining in which the firm has all the bargaining power.

3. The Firm's Choice of Contract

Workers and firms make decisions about whether to enter or leave matches by comparing the asset value of being in the match with the value of alternative opportunities available to them. Match values are affected by the contract offer strategies used by other firms, through the distribution function F, where $F(v)$ is the measure of firms offering contracts of initial value to the worker less than or equal to v. In this section we will analyse the optimal choice of contract for a single firm, taking $F(v)$ as exogenous. In the general equilibrium analysis, F will be the endogenously determined steady-state distribution of contract offers. Here we simply assume that F has the standard properties of a distribution function (it is non-decreasing and right-continuous) and has bounded support, as it must have in any equilibrium.

The joint continuation value to a worker and firm of a match with productivity p has constant maximum value $J^*(p)$, where J^* is a continuous and increasing function defined by the Bellman equation:

$$(\mu+\lambda) J^*(x) = x + \lambda E[\max(J^*(x), v); F] \tag{1}$$

The maximum initial value of the match is $J^*(p)$-s. The value to a worker of remaining in unemployment is $V_u = J^*(b) < J^*(p)$.

When a relationship is governed by a contract, the worker and firm optimise individually within the framework provided by the contract, so the maximum joint value $J^*(p)$-s is not necessarily achieved. Let $J_0(C) \leq J^*(p)-s$ be the initial joint value of the match with a contract C, and $V_0(C)$ be the initial value to the worker. The initial value to the firm is then J_0-V_0.

Now consider the firm's choice of contract offer when the exogenous probability of acceptance of an offer worth v to the worker is $P(v)$. Again, P will be endogenously determined in general equilibrium. Here we assume that P is non-decreasing, that $P(V_u\text{-})=0$, and that P is left-continuous above V_u; given the assumptions about workers' behaviour, these properties must hold in equilibrium. The firm's problem can be expressed:

$$\max_C \Pi(C) \equiv \big(J_0(C) - V_0(C)\big) P(V_0(C)) \tag{2}$$

3.1 Unrestricted Contract Choice

PROPOSITION 1: When contract choice is unrestricted, there exists an optimal contract for the firm with a constant wage $w = p$, and a fee $A^*=\text{argmax } (A-s)P(J^*(p)-A)$ paid by the worker on acceptance.

PROOF: The proposed contract has initial values $V_0 = J^*(p)-A^*$ to the worker and A^*-s to the firm. The expected payoff for the firm is $\Pi^*=(A^*-s)P(J^*(p)-A^*)$. With an arbitrary contract C:

$$\Pi(C) = \left(J_0(C) - V_0(C)\right)P(V_0(C))$$
$$\leq \left(J^*(p) - s - V_0(C)\right)P(V_0(C))$$
$$\leq \max_v \left(J^*(p) - s - v\right)P(v) = \max_A \left(A - s\right)P(J^*(p) - A) = \Pi^* \qquad \square$$

The intuition here is that by setting an acceptance fee A the firm can freely determine the worker's valuation V_0 without affecting the joint value of the match. So an optimal strategy is to maximise the joint value by ensuring that the worker receives the whole surplus ex-post and hence makes efficient quit decisions, then to use the fee to extract part of the surplus ex-ante. Effectively, the firm uses the fee to separate its recruitment strategy from its retention strategy.

It is clear that this is not the only contract that is optimal for the firm. First, it is straightforward to show that exactly the same effect[7] can be achieved with a contract without an initial fee, but with a lower wage $w=p-\mu A^*$ and a quitting fee A^* paid by the worker to the firm if he leaves. Second, depending on the distribution of offers by other firms, F, it may be possible to use a low initial wage instead of an acceptance fee. We consider such alternative contracts in 3.3 and 4. below.

Note that for the firm to make positive profit, it is necessary that $J^*(p)-s \geq J^*(b)$, and this condition depends on the distribution F. If the condition does not hold, the firm cannot do better than to offer a contact with $A=s$ and $w=p$, which would not be accepted by the worker.

Note also that for the contract in Proposition 1 to be optimal, the firm must be able to commit to paying the wage w in the event of employment. Without this the firm would

[7] That is, V_0 and J_0 are the same, and both parties make the same decisions about subsequent continuation of the match.

have an incentive to lower the wage ex-post, knowing that time will elapse before the worker receives an alternative offer.

3.2 Restricted Contract Choice

In practice, the firm may be restricted in its choice of contract instruments: use of entry fees or very low wages might be ruled out because workers are credit constrained, or by legislation; similarly quitting payments, as a form of bonding, may be illegal. We now examine the firm's choice of contract when it is unable to use any instantaneous fees, and there is a lower bound on the wages acceptable to workers. The firm can, however, make wages contingent upon the worker's tenure, T, in the match. Assume as before that the firm can commit to a future wage profile; since it learns nothing ex-post that affects its valuation of the relationship, it prefers to specify wages ex-ante, and we will suppose that the courts will enforce such contracts.

Thus we assume that the firm is able to choose any wage-tenure profile $w(T)$ satisfying the following conditions:

(i) $w(T)$ is continuous, except possibly at a finite number of points, at which it is right-continuous;

(ii) $w(T) \geq c$ for all T, where $c \leq b$;

(iii) $w(T) = \overline{w}$ for all $T \geq \overline{T}$, for some $\overline{T} \geq 0$.

c, the lower bound on wages, cannot be greater than the flow utility of unemployment b. Condition (iii), that the wage must eventually be constant, is not restrictive, but helps to ensure the existence of the value functions in what follows.

Consider a match for which a contract C corresponding to $w(T)$ has been signed and the match tenure is $T \geq 0$. Let $J(T;p;w)$ be its joint continuation value to the firm and worker, assuming that they will make individually optimal decisions in the future. Similarly let $V(T;w)$ be its value to the worker. The value of the match to the firm is J-V. Van den Berg (1990) establishes the existence and properties of the value functions for this type of non-stationary problem. Given the assumptions about the form of the contract, J and V are continuous and right-differentiable functions of tenure, T. For the worker and the firm to be willing to remain in the match requires:

$$J(T) \geq V(T) \geq V_u \tag{3}$$

While conditions (3) hold, the functions J and V satisfy:

$$(\mu + \lambda) J(T) = p + J'(T) + \lambda (J(T) - V(T)) F(V(T)) + \lambda E[\max(V(T), v); F] \tag{4}$$

$$(\mu+\lambda)V(T) = w(T) + V'(T) + \lambda E[\max(V(T), v); F] \tag{5}$$

The initial joint value of the match is $J_0(C) = J(0;p,w) - s$, and the worker's initial value is $V_0(C) = V(0;w)$.

In the unrestricted case in section 3.1, an optimal strategy involves setting the wage to maximise the joint value of the match and optimise retention, and an acceptance fee to optimise recruitment. This idea can be extended to cover the restricted case. We can try to separate the optimisation problem (2) into two steps as follows:

<u>Retention</u>: $$\bar{J}(v;p) = \max_{w(T)} J(0;p,w) \text{ subject to } V_0(C) = v \tag{6}$$

<u>Recruitment</u>: $$\max_{v}\left(\bar{J}(v;p) - s - v\right)P(v) \tag{7}$$

Problem (6) is to find a wage profile that maximises the joint value of the match, given that the contract must have a particular initial value v to the worker. Problem (7) is to choose this initial value optimally. To show that this separation is legitimate, we need to prove that (6) has a solution for any feasible initial value v.

A contract analogous to the "fee-plus-constant-wage" that is optimal in the unrestricted case would be a wage equal to c (the lowest acceptable wage) for some initial period, and subsequently equal to productivity. The lower is c, the more closely this contract can approximate the one with the entry fee. Contracts of this type, both in the restricted and unrestricted cases, will be referred to as *step-contracts*. We will show that step-contracts are optimal in the restricted case too.

Let $\tilde{V}(T;\tilde{T})$ be the value to the worker at tenure T of a step-contract in which the wage rises from c to p at tenure $\tilde{T}$. Then $\tilde{V}(T)$ satisfies:

$$\begin{aligned} &(\mu+\lambda)\tilde{V}(T) = c + \tilde{V}'(T) + \lambda E[\max(\tilde{V}(T), v); F] \qquad \forall T < \tilde{T} \\ &\tilde{V}(T) = J^*(p) \qquad \forall T \geq \tilde{T} \end{aligned} \tag{8}$$

$\tilde{V}(T)$ can be defined for negative as well as positive T, as the solution of the differential equation on the interval $(-\infty, \tilde{T}]$. Note that that $\tilde{V}(T;\tilde{T}) = \tilde{V}(T-a;\tilde{T}-a)$ for any constant a: the value of the contract depends only on tenure *relative* to the time when the wage steps up to p. For completeness, we also refer to the contract in which the wage remains at c for ever (corresponding to $\tilde{T} = \infty$) as a step-contract.

Intuitively, it is clear that the value of a step-contract increases as tenure approaches $\tilde{T}$. Since tenure relative to this moment is what matters, it follows that by varying $\tilde{T}$ it is possible to choose any initial value for a step-contract.

LEMMA 1: $\tilde{V}(T;\tilde{T})$ is strictly increasing in T on $(-\infty, \tilde{T}\,]$ and $\tilde{V}(T) \to J^*(c)$ as $T \to -\infty$.
PROOF: See appendix.

The wage and value profiles for a step-contract are shown in Figure 1.

LEMMA 2: For any $v \in [J^*(c), J^*(p)]$, there is a uniquely-defined step-contract $S(v,p,c)$ with initial value v to the worker.

PROOF: The initial value of a step-contract $\tilde{V}(0;\tilde{T}) = \tilde{V}(-\tilde{T};0)$. Hence, using Lemma 1, $\tilde{V}(0;\tilde{T})$ is continuous and strictly decreasing in $\tilde{T}$, with $\tilde{V}(0;0) = J^*(p)$ and $\tilde{V}(0;\tilde{T}) \to J^*(c)$ as $\tilde{T} \to \infty$. So, for any $v \in (J^*(c), J^*(p)]$, there exists a unique $\tilde{T}$ such that $\tilde{V}(0;\tilde{T}) = v$. Otherwise, if $v=J^*(c)$, the contract has $w=c$ for ever. ∎

So the firm can always find a step-contract for any feasible initial value v: workers will not accept contracts of lower value than $J^*(c)$, and the firm would make negative profit with a value higher than $J^*(p)$. Moreover, for given v, the step-contract is optimal:

LEMMA 3: For any $v \in [J^*(c), J^*(p)]$ the step-contract $S(v,p,c)$ solves problem (6).
PROOF: See appendix.

The intuition for this result is that with a step-contract the value to the worker rises as fast as possible with tenure, for a given initial value. This minimises the probability that the worker will accept an alternative offer of lower value than the current match, and hence maximises the joint value of the match. It is now trivial to prove:

PROPOSITION 2: If there is any acceptable contract that gives non-negative profit for the firm, there exists a step-contract $S(v^*, p,c)$ that is optimal.
PROOF: Let C be a contract with $J_0(C) \geq V_0(C) \geq J^*(b)$. From Lemma 2, $\bar{J}(v)$ exists for any $v \in [J^*(c), J^*(p)]$. Let $v^* = \underset{J^*(c) \leq v \leq J^*(p)}{\arg\max} \left(\bar{J}(v;p) - s - v\right)P(v)$. Then:

$$\Pi(C) = \left(J_0(C) - V_0(C)\right)P(V_0(C)))$$

$$\leq \left(\bar{J}(V_0(C)) - s - V_0(C)\right)P(V_0(C))$$
$$\leq \left(\bar{J}(v^*) - s - v^*\right)P(v^*) = \Pi(S(v^*, p, \Sigma))$$

Hence C cannot give higher payoff than this step-contract. ∎

3.3 A General Equilibrium Result

We have now established that there is always, whether or not the form of contracts is restricted, a step-contract that is optimal for the individual firm. However, there may be alternative contracts that are also optimal. The alternative contracts available to the firm are, in the restricted case, other wage profiles $w(T)$, and in the unrestricted case contracts consisting of an initial fee A and a wage-tenure profile $w(T)$[8]. For the individual firm, an alternative contract can be optimal only if its initial value and retention properties are the same as those of an optimal step-contract. The following result holds in both the restricted and unrestricted cases:

LEMMA 4: If C is an optimal contract with initial value to the worker $V_0(C)$, and value profile $V(T)$, then the step-contract with the same initial value, and value profile $\tilde{V}(T)$, is also optimal, and the probability of the worker leaving is the same at all times in both contracts: $F(\tilde{V}) = F(V)\ \forall T$.

PROOF: See appendix.

This result suggests that if we are interested in how the labour market allocates resources in equilibrium, we can restrict attention to equilibria in which firms use step-contracts. Proposition 3 confirms this intuition:

PROPOSITION 3: For any Nash equilibrium in contract offers, there exists an equivalent Nash equilibrium in offers of step-contracts, in which all turnover decisions are identical.

PROOF: See appendix.

We now apply this result to analyse equilibrium turnover, first when all firms are identical (Section 4), and then when there are two types of firm (Section 5).

4. Homogeneous Firms

Suppose that all firms have identical productivity p and recruitment costs s, satisfying $p-\mu s>b$ (employment has strictly higher social value than unemployment).

[8] Other lump-sum payments, such as a quitting fee, are possible but redundant.

Burdett and Mortensen show, for the case of no turnover costs (s=0), that if firms are restricted to offering a constant wage there is no equilibrium in which they all offer the same wage. It is easy to see why: first, if all firms offer a particular wage below p, a firm offering a slightly higher wage will attract a much larger supply of workers, while hardly losing any of its profit on each worker; and if all firms offer a wage p (making zero profit), a firm offering a slightly lower wage will still attract unemployed workers, and will make positive profit. The unique equilibrium has a continuous distribution of wages. Firms effectively randomise over wages; they obtain equal payoffs from a low wage, with correspondingly high turnover, and from a higher wage and lower turnover. Burdett and Mortensen do not consider turnover costs; however, the argument above extends to this case: even when s>0, there is no equilibrium in which firms offer the same wage, so any equilibrium must involve job-to-job turnover, which is necessarily inefficient.

But if firms are able to vary wages with tenure, there can be no equilibrium of this type. When the wage is constant over time, the value of the contract to the worker remains constant over time also. By varying the wage the firm can make an offer that has the same initial value to the worker as any constant wage, but an increasing value while he remains with the firm. This means that, for given strategies of other firms, the firm can maintain the same level of recruitment, while increasing the retention rate. The effect is to eliminate turnover between jobs.

To prove this we can, by Proposition 3, restrict attention to equilibria in which firms offer step-contracts. First note that if the unemployment rate is u, workers enter unemployment at rate $\mu(1-u)$ and leave at a rate $\leq\lambda u$, depending on offers. So in steady state there is a positive measure of unemployed workers $u \geq \mu / (\mu + \lambda)$. If firms can make positive profits in equilibrium, all firms will make offers that are acceptable to the unemployed, so workers will leave unemployment at rate λu, and $u = \mu / (\mu + \lambda)$.

4.1 Equilibrium in Unrestricted Step-Contracts

When firms are able to use entry fees, it is obvious that there can be no turnover in equilibrium. A step-contract has $V(T)= J^*(p)$ for all T. So no firm can make an offer acceptable by employed workers, and $V(T)=p/\mu$. It follows immediately that the unique equilibrium is for firms is to use an entry fee so that $V_0=V_u$, attracting unemployed workers only, in which case $V_0=V_u= b/\mu$. Workers stay until retirement, and firms capture all the match rent.

4.2 Equilibrium in Restricted Step-Contracts

The zero-turnover result is less obvious with restricted step-contracts, since the firm is vulnerable to outside offers during the initial period. First note that in an equilibrium in which all firms face the same restrictions, all workers will have step-contracts with the same *c* and *p*. An employee's reservation value depends only on his tenure *relative* to the moment when the step-up occurs, $R \equiv T - \tilde{T}$. So we can describe all workers by their current relative tenure, and look for an equilibrium in offers of relative tenure.

Define R_u as the value of relative tenure at which the step-contract has the same value as unemployment[9], $\tilde{V}(R_u;0) = V_u$. Then define *H* as the steady-state distribution of offers, $H(R) \equiv F(\tilde{V}(R;0))$, and *G* as the probability that a worker will accept an offer of *R*, $G(R) \equiv P(\tilde{V}(R;0))$. *H* is increasing and right-continuous, but may have mass points where $H(R) > H(R-)$. $G(R)$ is the stock of workers with current relative tenure less than or equal to *R*, so we can evaluate the steady-state *G* for given *H* by equating the flows of workers entering and leaving this stock:

LEMMA 5: The steady-state distribution of tenure $G(R)$ is continuous and has right-derivative:

$$G'(R) = \mu - (\mu + \lambda(1 - H(R))G(R) \qquad \forall R \geq R_u$$

and $G(R_u) = \dfrac{\mu}{\mu + \lambda(1 - H(R_u))} = u$.

PROOF: See Appendix.

Now let $Y(R)$ be the firm's match continuation value. *Y* is also continuous and right-differentiable, and from (4) and (5):

$$Y'(R) = (\mu + \lambda(1 - H(R)))Y(R) - (p - c) \ \forall R < 0 \text{ and } Y(0)=0 \tag{9}$$

No firm will offer $R > \bar{R}$, which is the point at which $Y=s$ if the worker never quits[10], so $H(\bar{R}) = 1$. The firm's payoff from an offer *R* is: $\Pi = (Y(R) - s)G(R)$; using Lemma 5 and (9) we can find the optimal choice of *R*. In the proof of Proposition 4, we show there can

[9] In this section we will assume $b<c$, so that R_u is finite. The same results hold for $b=c$, with a slight adjustment to the proofs.

[10] The unique $\bar{R}$ is found by solving (9) with $H \equiv 1$ for *Y* and then solving $Y(R)=s$.

be no-job-to-job turnover by proving that in any plausible equilibrium, H has a single mass point of mass one. We then prove the existence of equilibria of this type.

PROPOSITION 4: Suppose all firms are identical, with p-$\mu s>b$, and the same restrictions, c, on contracts. Then:

(i) In any equilibrium in contract offers in which firms make strictly positive profits, all offers have the same initial value and there is no job-to-job turnover.

(ii) It is an equilibrium for all firms to offer a step-contract of the same initial value V_0 if and only if $V_0 \in [\underline{V}, \overline{V}]$ where:

$$\mu\underline{V} \equiv \max\left(b, \frac{\lambda(p-\mu s)+\mu c}{\lambda+\mu}\right) \text{ and } \mu\overline{V} \equiv \min\left(p-\mu s, \max\left(b, \frac{\lambda p+\mu c}{\lambda+\mu}\right)\right)$$

(iii) There exist equilibria with job-to-job turnover if and only if p-$c<(\lambda+\mu)s$. A fraction $\alpha \equiv (p-c-\mu s)/\lambda s$ of firms offer a contract of value to the worker $V_H=p/\mu-s$, a mass of firms of positive measure $\beta\leq 1-\alpha$ offer $V_L \in [V_u,V_H)$, and the set of firms making any other acceptable offer is of measure zero.

PROOF: See appendix.

Proposition 4 means that efficient equilibria with zero turnover exist for all parameter values, with a unique initial value if either s=0 or c is sufficiently low.

With turnover costs and credit constraints, firms face particular problems: they must incur the initial cost, and need to retain the worker for a long time at a wage c in order to recoup the cost. In this case there are multiple equilibria, which when s and c are both high may include inefficient equilibria with turnover, although we may argue that these are implausible. They require a mass of exactly α firms to make an offer giving the whole match surplus to the worker. α is such that firms making a lower offer also make zero profits because the value to the firm of a potentially longer period at a low wage is exactly offset by the probability that the worker will leave. These equilibria are unstable in the sense that a small increase in the mass offering V_H would lead all firms to deviate to V_H, which would also be an equilibrium. Similarly a decrease in the mass at V_H would lead all firms to deviate to an offer of V_L or lower.

Proposition 4 completely characterises the equilibria in step-contracts. When restrictions are low, or absent, the equilibrium offer has value $V_0=b/\mu$. When c is higher, the equilibrium offer is higher too. Firms cannot use very low wages, and need to raise the wage quickly to deter turnover, so the contract has higher value to workers: credit constraints effectively give workers some bargaining power, and thence allow them to capture some of the match rent.

Solving equation (4) we can obtain the length of the initial low wage period: $\mu\tilde{T} = \ln(p-c) - \ln(\mu V_0 - c)$. It can then be verified that credit constraints reduce the "slope" of equilibrium wage profiles, where "slope" means the jump in the wage, relative to the waiting time: $(p\text{-}c)/\tilde{T}$. Equally intuitively, a reduction in frictions (an increase in the meeting rate λ) increases the equilibrium V_0, and hence reduces the waiting time and increases the slope.

Note also that for any finite c, the equilibrium offer $V_0 \to p/\mu\text{-}s$ as $\lambda \to \infty$. This is the competitive labour market outcome, in which workers capture the whole match surplus. This may be contrasted with the case of unrestricted contracts, when for all values of λ the equilibrium step-contract is a wage equal to productivity and a fee delivering the whole surplus to the firm[11].

4.3 Alternative Equilibrium Contracts

Finally for the homogeneous case, it is instructive to consider what alternatives to step-contracts may be used in equilibrium. In fact there is a wide range of possible equilibrium contracts; what these have in common is that their value to the worker rises with tenure, sufficiently steeply to deter turnover. In order to describe the equilibrium contracts, define:

$$W(T;w) \equiv \int_T^\infty e^{-\mu(s-T)} w(s)ds \tag{10}$$

and

$$\mu U(T;V_0) \equiv \frac{\mu^2 V_0 + \lambda(p-\mu s)(1-e^{-\mu T})}{\mu + \lambda(1-e^{-\mu T})} \tag{11}$$

$W(T;w)$ is the value at tenure T of a wage stream w. The function $U(T)$, with $U(0)=V_0$, provides a lower bound for the value profile of contracts that can be used in equilibrium.

[11] With unrestricted contracts and an infinite meeting rate there is a continuum of additional equilibria. A wage equal to productivity together with any fee that shares the rent between worker and firm can be an equilibrium contract. This indeterminacy happens because there are fixed numbers of homogenous firms and workers and a positive rent even when frictions disappear. When there is perfect competition between workers for jobs, and between firms for workers, the “competitive equilibrium” does not determine the sharing of the rent.

Proposition 5[12] states that any wage profile that has initial value V_0, and subsequently higher value than U, is an alternative to the step-contract with initial value V_0.

PROPOSITION 5: If all firms are identical, with p-$\mu s>b$, and the same restrictions, c, on contracts, and $V_0 \in [\underline{V}, \overline{V}]$, any wage profile satisfying the following conditions supports a symmetric Nash equilibrium: $c \leq w(T) \leq p \ \forall T \geq 0$; $W(0;w)=V_0$; and $U(T;V_0)<W(T;w) \ \forall \ T>0$.
PROOF: See appendix.

Figure 2(a) illustrates the region within which the value of the wage stream W must lie, for a case when c is sufficiently low that equilibrium contracts have initial value V_u. The lower boundary of the region is given by the function U. To see the intuition behind Proposition 5, suppose that all firms are using a particular contract C_0 within the shaded region. Because the value rises with tenure, employed workers will not accept alternative offers of C_0. Hence there is no turnover, and there is a steady state distribution of job tenure determined by the rates at which workers meet firms and retire. The value-tenure profile is sufficiently steep to make the distribution of reservation values of employed workers unfavourable for firms considering deviating from C_0. There is a low probability of meeting an employed worker who will accept a low offer. So it is better to use C_0, which recruits unemployed workers cheaply and keeps them forever, than to offer contracts of value high enough to attract significant numbers of employed workers.

The upper boundary of the equilibrium region corresponds to the most steeply-rising contract that is possible given the lower bound on the wage, which is the step-contract. Figure 2(b) shows the wage profiles corresponding to the two boundaries. Note that the indeterminacy in wage profiles arises from the assumption of risk neutrality. With risk-averse workers the flatter wage profile corresponding to U would be preferred to the step-contract. A reasonable conjecture would be that in this case there would be a unique equilibrium contract flat enough to be attractive to workers but steep enough to deter turnover.

5. Heterogeneous firms

In the homogeneous case tenure-related contracts enhance the efficiency of matching in the labour market by eliminating job-to-job turnover. When firms differ in productivity, the use of contracts with tenure-related payments has different implications.

[12] Proposition 5 is proved for restricted contracts, but the alternative contracts for the unrestricted case are

For socially efficient job choices it is necessary that there is an increasing relationship between the worker's valuation of a contract and the present value of his productivity. Burdett and Mortensen's model, with constant productivity, no turnover costs, and firms restricted to constant wages, has this property. When there are high and low productivity firms, each type randomises over a distribution of wages, but these distributions are non-overlapping: a worker in a low productivity firm who meets a high productivity firm will always receive a better offer.

But from Propositions 1 and 2 we know that step-contracts, in which the worker's value rises with tenure, are optimal. With heterogeneous firms this will lead, in general, to inefficient matching, because a worker's response to alternative offers will depend on his tenure in the current match, although this has no effect on his relative productivity. Thus, for example, a worker with high tenure in a low-productivity match may refuse an offer from a high-productivity firm because of its low initial value.

In this section we will examine equilibrium turnover when there are two types of firms; half of them with high productivity, and half with low productivity:

$$(p,s) = \begin{cases} (p_1, s_1) \\ (p_2, s_2) \end{cases} \text{with probability } \tfrac{1}{2} \quad \text{where} \quad \begin{array}{l} b < p_1 - (\mu + \tfrac{1}{2}\lambda)s_1 \\ \text{and} \quad p_1 < p_2 - \mu s_2 \end{array} \tag{12}$$

The inequalities ensure that it is socially optimal for unemployed workers to move into employment with either type of firm, and for workers employed in 1-firms to move to 2-firms when a meeting occurs. Applying Proposition 3, it is sufficient for the analysis of turnover to look at equilibria in step-contracts.

5.1 Equilibrium in Unrestricted Contracts

With unrestricted step-contracts, the wage is always equal to productivity, so there can be no movement of workers between firms with the same productivity. The inequalities in (12) ensure that all firms can make positive profit, and $u=\mu/(\mu+\lambda)$. 1-firms recruit only from unemployment, and offer contracts of initial value V_u. The only job-to-job turnover that is possible in equilibrium is from 1-matches to 2-matches. Furthermore, *if* such turnover occurs, it must be achieved by 2-firms offering contracts of initial value equal to $J^*(p_1)$ (or rather, infinitesimally more). For workers in 1-firms earn a wage equal to p_1 and hence will not accept less, and 2-firms cannot both attract more workers and

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revealed by letting $c \to -\infty$.

make positive profit with a higher offer. Alternatively, 2-firms may find it better to offer a contract that only attracts unemployed workers.

Thus, there are two possible types of equilibrium: a separating equilibrium, in which 1-firms offer contracts of initial value V_u and 2-firms offer contracts of initial value $J^*(p_1)$; and a pooling equilibrium, in which both types offer contracts of initial value V_u. Clearly, the first of these has socially-optimal matching of workers to firms, whereas in the second, turnover is inefficiently low. Proposition 5 shows that either of these equilibria can exist, depending on the parameters, and identifies the conditions required for efficient turnover.

PROPOSITION 6: When there are two types of firms, as defined by (12), and contract choice is unrestricted, an efficient full-turnover equilibrium exists if and only if:

$$\frac{(p_2 - \mu s_2 - b)}{(p_1 - b)} \geq 1 + \frac{2\mu}{\lambda} \tag{13}$$

Otherwise there is a zero-turnover equilibrium in which workers enter firms from unemployment and do not leave.

PROOF: See appendix

The inequality (13) can be compared with the condition for job-to-job turnover to be socially optimal, which can be written as:

$$\frac{p_2 - \mu s_2 - b}{p_1 - b} \geq 1 \tag{14}$$

In equilibrium, job-to-job turnover occurs only if the value of the productivity profile in 2-firms exceeds that in 1-firms by a positive margin, that depends on the parameters of the matching process. The outcome with no job-to-job turnover occurs when the meeting rate λ is low (relative to the retirement rate μ). In this case it is less worthwhile for high-productivity firms to try to attract workers from low productivity firms. They find it more profitable to offer contracts of low initial value, attracting unemployed workers only, and keeping them until retirement.

Note that turnover inefficiency is not related to the presence of turnover costs. The equilibrium condition (13) involves a comparison of the same measures of productivity as the social optimum condition (14). When there are costs of turnover, a contract with an entry fee and a wage equal to productivity is an appropriate response; what Proposition 6 tells us is that firms will "over-use" this form of contract, to reduce turnover below the socially desirable level.

5.2 Equilibrium in Restricted Contracts

When firms are not able to charge entry fees or use very low initial wages, their ability to extract rent is reduced, as we saw in section 4.2. Since the turnover inefficiency in Proposition 5 arises from the firms' use of entry fees to extract rent, we might expect that restrictions on their ability to do this would enhance the efficiency of matching.

In the unrestricted case, equilibria had either efficient full-turnover, or zero-turnover. With restricted contracts it is also possible to have equilibria with partial-turnover, in which higher productivity firms offer contracts of value high enough to attract workers of low tenure in low productivity firms, but not those of high tenure.

The full model is algebraically complex in this case, so we will simplify by assuming, first, that there are no turnover costs, and second that the restrictions take the severest possible form: $c=b$. For an efficient equilibrium, it must be the case that all firms of the same type offer the same contract, and that the contract offered by a 2-firm is sufficient to attract any worker in a 1-firm contract. Conversely, zero-turnover occurs if and only if all firms offer the same contract. To find these equilibria, and intermediate ones, we can look for an equilibrium in which three step-contracts are offered: all 1-firms offer the same contract of initial value V_{01}, a fraction $f\in[0,1]$ of 2-firms offer a contract with $V_{02} \geq J^*(p_1)$, sufficient to attract all 1-firm workers, and the remaining 2-firms offer a contract of lower value $V_{01} \leq V_{0L} < J^*(p_1)$.

There is a unique step-contract for an i-firm with initial value V_{01}: $S_i \equiv S(V_{01}, p_i, c)$. Furthermore, the value functions for these two contracts evolve identically while the wage remains at c. Hence we can describe the choice facing an i-firm as a choice of an offer of tenure within S_i, and all workers by their effective tenure in these contracts. If the wage rise in S_i occurs at tenure $\overline{T}_i$, then $\overline{T}_1 < \overline{T}_2$. An offer T=0 is a contract of value V_{01}, and an offer $0<T\leq \overline{T}_1$ would attract any worker with current effective tenure less than T in either of these contracts. A 2-firm could offer a contract with $T> \overline{T}_1$, which would attract all workers in 1-firms, as well as those in 2-firms with lower tenure.

By analysing the equilibria in tenure choices in such contracts, we obtain the following results:

PROPOSITION 7: When there are two types of firms, as defined by (12), no turnover costs ($s_1 = s_2 = 0$), and firms are restricted to $w \geq b$:

(i) there is no zero-turnover equilibrium;

(ii) an efficient full-turnover equilibrium exists if and only if:

$$\frac{p_2 - b}{p_1 - b} \geq \frac{2\mu + \lambda}{2\mu A^2 + \lambda} > 1 \text{ where } A \equiv \frac{4\lambda(\mu + \lambda)}{\lambda^2 + 4(\mu + \lambda)^2} \in (0,1) \; ; \tag{15}$$

(iii) otherwise there is a partial-turnover equilibrium in which some, or all, high productivity firms offer contracts that are not acceptable to workers of high tenure in low productivity firms.

PROOF: See Appendix

The critical value of relative productivity above which full turnover occurs is lower than in the unrestricted case. Thus, the restrictions on contracts facilitate efficient matching: full turnover occurs for a wider range of parameters, and even where it does not, there is partial turnover. As in the unrestricted case, the outcome depends on relative productivity and on the meeting rate relative to the retirement rate, λ/μ. Figure 3 shows how conditions (14) and (15) partition the parameter space, to compare equilibrium turnover in the two cases.

6. Discussion

6.1 Relaxing Some Assumptions

In order to obtain a straightforward characterisation of the equilibrium contracts we have made some very simple technological assumptions: that firms have constant returns to scale, and that workers are identical in their utility from unemployment. Together these imply that the match rent *p-b* is independent of the employment rate. In principle there is no difficulty in relaxing either, or both, of these assumptions. Then, the equilibrium contracts are still upward sloping. Due to the existence of frictions the value of unemployment to the marginal employed worker lies strictly below the value of his productivity, and contracts slope upwards between these two values. Assuming that workers' reservation utilities are not observed by firms, and they are still therefore required to offer all workers the same contract, there would be an additional inefficiency: unemployment would be higher than the frictional level because some workers would never accept employment offers despite potential gains from trade. Burdett and Mortensen demonstrated this type of unemployment inefficiency for the wage-posting equilibrium with heterogeneous workers.

A critical feature of the model is that the firm must offer the same contract to every worker. This follows from the assumption that the firm cannot observe anything about the worker that would help it to assess his reservation value. Since it may seem implausible that the firm cannot even observe whether the worker is employed, consider the effect of supposing that firms could make different offers to employed and unemployed workers. In the homogeneous case with unrestricted contracts there would still be no turnover in equilibrium, but the only equilibrium contract would have an entrance fee and wage equal to productivity. With restricted contracts, upward-sloping contracts would be used but there would be some (inefficient) turnover in equilibrium. Unemployed workers would obtain lower value offers than employed workers and turnover would occur when an employed worker met another firm soon after recruitment from unemployment, while the value of his contract was still low.

In the heterogeneous case, knowledge of employment status would destroy the pooling equilibrium of Proposition 5, and turnover would be efficient. However, precisely the same kind of turnover inefficiency would reappear if there were three or more (rather than two) types of firm, provided that firms were unable to observe the productivity of a worker's current match. The implications of these results are discussed further below.

6.2 Interpretation of the Results

The equilibrium contracts in the model of this paper can be interpreted as internal labour market strategies. Matches have specific value because of the time lags in finding alternative partners, and this provides an incentive to adopt long-term contracts. Firms offer contracts that are privately efficient: they maximise the joint value of a match. The consequence is low labour market turnover, with wages increasing with tenure and some or all workers remaining in the same firm throughout their careers. Thus, the model can be interpreted as establishing that internal labour markets are likely to develop when:

- there is little uncertainty about productivity[13];
- workers can move from firm to firm but receive information slowly about other jobs;
- firms have little information about the alternative opportunities available to workers.

[13] If, on the other hand, productivity were subject to shocks, it would not be optimal to pay high wages ex-post, and the outcome would be quite different.

In these circumstances it is optimal for firms to offer contracts whose value rises with tenure, and they can do this even when workers are severely credit constrained.

Although these strategies are privately efficient, they may result in too little turnover for the socially efficient matching of workers to jobs. Using entry fees or other forms of tenure-related contract is an effective means of rent-extraction for the firm. It is a form of monopsonistic price discrimination, competing on different terms in the markets for new and existing workers. In the full information case (section 2.1) the firm can discriminate perfectly, extracting all the rent with no adverse effect on aggregate output. With asymmetric information and hence anonymous contracts, in the homogeneous case (section 4) it succeeds in discriminating perfectly in equilibrium, and again, the outcome is efficient. But when firms differ in productivity (section 5), they are no longer able to price-discriminate perfectly. The best that they can achieve is third-degree price discrimination in which different wages are used in the internal and external markets, and the effect is to give the wrong signals to workers, reducing aggregate output.

Furthermore, conditions or institutions that restrict the firms' ability to price-discriminate (imperfectly) are beneficial, in that they promote efficient matching. When workers are unable to accept contracts involving very low wages because of credit-constraints, firms cannot use steeply-rising contracts. Effectively, the workers' bargaining power is increased, and firms have less monopsony power. A minimum wage, and legal restrictions on bonding, have similar effects. We cannot, however, conclude that rising wage-tenure contracts are always indicative of monopsony power. For efficient matching, positive turnover costs (or other forms of specific capital that increase productivity with tenure) must be reflected in the contract.

More generally, the possibility of price discrimination in frictional labour market models suggests an efficiency rationale for social conventions of fairness which dictate the payment of "a wage for the job" rather than allowing firms to tailor contracts more closely to individual circumstances (where these do not affect productivity). In section 6.1 we briefly considered the effects of allowing firms to offer different contracts to employed and unemployed workers. If this enables firms to price-discriminate perfectly, it can improve efficiency; when it does not so, providing further opportunities for firms to engage in third-degree price discrimination can worsen the outcome. Individual contracts are privately and socially efficient when firms are fully informed about the individual. Otherwise their attempts to exploit limited information can distort price signals and reduce aggregate welfare.

References

ANDREWS, M., S. BRADLEY, and R.UPWARD (1999)“Estimating youth training wage differentials during and after training”, *Oxford Economic Papers*, 51:3, 517-44.

BURDETT, K. and D. T. MORTENSEN (1998) “Equilibrium Wage Differentials and Employer Size”, *International Economic Review*, 39(2): 257-76.

DIAMOND, P. (1971) “A Model of Price Adjustment”, *Journal of Economic Theory*, 3, 156-68

DOERINGER, P. and M. PIORE (1971) *Internal Labour Markets and Manpower Analysis*, Lexington, Mass: Heath.

HALL, R. E. and E. P. LAZEAR (1984) “The Excess Sensitivity of Quits and Layoffs to Demand”, *Journal of Labor Economics*, 2:2, 233-57

HASHIMOTO, M. (1981) “Firm-Specific Human Capital as a Shared Investment”, *American Economic Review*, 71:3, 475-82

LAZEAR, E. P. (1981) "Agency, Earnings Profiles, Productivity, and Hours Restrictions", *American Economic Review*, 71(4), 606-20.

MANNING, A. (1993) “Endogenous Labour market Segmentation in a matching Model”, *Centre for Economic performance Discussion Paper* 126, London School of Economics

MANNING, A. (2000) “Pretty vacant: recruitment in low-wage labour markets”, *Oxford Bulletin of Economics and Statistics* (forthcoming)

MORTENSEN, D. T. and PISSARIDES, C. (1999) “New developments in models of search in the labour market” in O. Ashenfelter and R. Layard (eds.) *Handbook of Labor Economics*, v. 3B, Amsterdam:Elsevier.

MORTENSEN, D. T. and T. VISHWANATH (1993) “Information Sources and Equilibrium Wage Outcomes”, in Bunzel, H., Jensen, P. and Westergard-Nielsen, N. (eds.) *Panel Data and Labor Market Dynamics*, Amsterdam: North-Holland..

PISSARIDES, C. (1990) *Equilibrium Unemployment Theory*, Oxford: Blackwell.

SALOP, S. (1979) "A Model of the Natural Rate of Unemployment", *American Economic Review*, 69, 117-25.

SHIMER, R. (1996) “Contracts in a Frictional Labor Market”, mimeo, Massachusetts Institute of Technology

ULMAN, L. (1992) “Why should human resource managers pay high wages?” *British Journal of Industrial Relations*, 30:2, 177-212.

WACHTER, M.L. and WRIGHT, R.D. (1990) “The Economics of Internal Labour Markets”, *Industrial Relations*, 29:2, 240-262.

VAN DEN BERG, G.J. (1990) “Non-stationarity in job search theory”, *Review of Economic Studies*, 57:2, 255-77.

Appendix

LEMMA A1: If v is a random variable with distribution function F, the function $\Gamma(x) \equiv (\mu+\lambda)x - \lambda E\left[\max(x,v);F\right]$ is strictly increasing in x.

PROOF: For $x_1 > x_2$, write $E_i = E[\max(x_i, v)]$. Then

$$\begin{aligned} E_2 &= x_2 F(x_2) + E\left[v \middle| v > x_1\right](1-F(x_1)) + E\left[v \middle| x_2 < v \le x_1\right](F(x_1)-F(x_2)) \\ &= x_2 F(x_2) - x_1 F(x_1) + E_1 + E\left[v \middle| x_2 < v \le x_1\right](F(x_1)-F(x_2)) \end{aligned}$$

Since $E\left[v \middle| x_2 < v \le x_1\right] > x_2$, $E_1 - E_2 < (x_1 - x_2)F(x_1)$.

Then $\Gamma(x_1) - \Gamma(x_2) = (\mu+\lambda)(x_1-x_2) - \lambda\left(E_1 - E_2\right)$

$$> (\mu + \lambda(1-F(x_2)))(x_1 - x_2) > 0$$

∎

PROOF of LEMMA 1: Write (8) as: $\tilde{V}'(T) = \Gamma(V(T)) - c$ where Γ is defined in Lemma A1. $\tilde{V}(T) > J^*(c)$ and $\Gamma(J^*(c)) = c$. By Lemma A1, $\tilde{V}'(T) > \Gamma(J^*(c)) - c = 0$. $\tilde{V}(T)$ is bounded below, continuous and increasing, so as $T \to -\infty$, $\tilde{V}'(T) \to 0$ and hence $\Gamma(V(T)) \to c$, and since Γ is strictly increasing, $\tilde{V}(T) \to J^*(c)$. ∎

PROOF of LEMMA 3: Let $J(T)$ be the joint value and $V(T)$ be the value to the worker of an arbitrary contract with wage profile $w(T)$, where $V(0) = v \in [J^*(c), J^*(p)]$. Let $\tilde{V}(T)$ and $\tilde{J}(T)$ be the corresponding values for the step-contract $S(v,p,c)$, with step-up point $\tilde{T}$.

For $T \ge \tilde{T}$, $\tilde{V}(T) = J^*(p) \ge V(T)$.

For $0 \le T < \tilde{T}$, $\tilde{V}' = \Gamma(\tilde{V}) - c$ and $V' = \Gamma(V) - w(T)$. By Lemma A1, whenever $\tilde{V} \ge V$, $\tilde{V}' \ge V'$. But $\tilde{V}(0) = V(0)$. Hence $\tilde{V}(T) \ge V(T)\ \forall T$.

Compare $J(T)$ and $\tilde{J}(T)$. For $T \ge \tilde{T}$, $\tilde{J}(T) = J^*(p) \ge J(T)$. Using (4), for all T:

$$\begin{aligned}(\mu+\lambda)(\tilde{J} - J) = (\tilde{J}' - J') + \lambda\tilde{J}F(\tilde{V}) - \lambda J F(V) - \\ \lambda E[v \mid V < v < \tilde{V}](F(\tilde{V}) - F(V))\end{aligned} \qquad \text{(A1)}$$

Since $V \le \tilde{V} \le \tilde{J}$

$$(J' - \tilde{J}') \ge (\mu + \lambda(1-F(V)))(J - \tilde{J})\ \ \forall T \qquad \text{(A2)}$$

Then if $(J - \tilde{J}) > 0$ at any point T_1, $(J - \tilde{J}) > 0\ \forall T > T_1$. But $J(\tilde{T}) \le \tilde{J}(\tilde{T})$. Hence $(J - \tilde{J}) \le 0\ \forall T$, and in particular $J(0) \le \tilde{J}(0)$. ∎

PROOF of LEMMA 4: For the restricted case we know from Lemma 3 that $\tilde{V}(T) \geq V(T)$ and $\tilde{J}(T) \geq J(T)\ \forall T.$ For the unrestricted case this also holds since $\tilde{V}(T) = \tilde{J}(T) = J^*(p)\ \forall T.$ Hence the step-contract is optimal and $\tilde{J}(0) = J(0).$ In both cases $\tilde{J}$ and J satisfy (A1) and (A2). From (A2), if $(J - \tilde{J}) \geq 0$ at any point T_1, $(J - \tilde{J}) \geq 0$ $\forall T > T_1$. Hence $J(T) = \tilde{J}(T)$ and $J'(T) = \tilde{J}'(T)\ \forall T$.

Substituting in (A1) $\Rightarrow F(\tilde{V}) = F(V)\ \forall T.$ ∎

PROOF of PROPOSITION 3: Suppose that, in a Nash equilibrium, $F(v)$ is the distribution of offers and $P(v)$ is the probability of acceptance. If F and P do not change, we know that every firm is indifferent between its equilibrium contract choice and the step-contract of the same initial value. Suppose that every firm making an acceptable offer switches to the equivalent step-contract. We will show that this is also an equilibrium.

Since the step-contracts have the same initial values, F is unchanged and so is $\bar{J}(v).$ However, P may change to $\tilde{P}$ since employed workers may have a higher reservation value with step-contracts. (V_u is unchanged). So $\tilde{P}(v) \leq P(v)$. To show that this is an equilibrium, we show that there is no incentive for a positive measure of firms to change their initial offers. Let $\chi = \{v: F(v) > F(v-\varepsilon)\ \forall \varepsilon > 0\}.$ Let η denote the set of firms who make an initial offer $v \notin \chi$. By definition, the measure of η is zero[14].

Now take any $v \in \chi$. Suppose an employed worker who had a value v' in the original equilibrium now has a value $\tilde{v}'$. Then $\tilde{v}' \geq v'$.

If $v' \geq v$ then $\tilde{v}' \geq v$.

If $v' < v$ then by definition of χ, $F(v') < F(v).$ From Lemma 4 $F(\tilde{v}') = F(v')$ so $F(\tilde{v}') < F(v)$ and $\tilde{v}' < v.$

Hence the measure of workers with reservation values less than v remains the same; that is, $\tilde{P}(v) = P(v)\ \forall v \in \chi.$

So any firm outside η earns the same profit as in the original equilibrium if it uses the same initial value v_0. If it changed its offer to v, it would get $\tilde{\Pi}(v) = (\bar{J}(v) - s - v)\tilde{P}(v) \leq (\bar{J}(v) - s - v)P(v) \leq \Pi(v_0) = \tilde{\Pi}(v_0)$ so v_0 is still optimal. So no firm outside η has an incentive to change its initial offer. Since η is of measure zero, this must be an equilibrium. ∎

[14] If firms of positive measure make offers at a point or over an interval, then by definition F increases there.

PROOF of LEMMA 5: When employed workers have step contracts, only unemployed workers accept offers of R_u so $G(R_u)=u$. Equating flows in and out of the stock of unemployed workers gives the expression for u. For $R \geq R_u$, $G(R+dt)$ is the stock of unemployed workers and employed workers with current tenure less than $R+dt$. The number entering this stock in a time period of length dt is $(1- G(R+dt))\mu dt$, the workers with higher tenure who are replaced by unemployed workers. In steady state this is equal to the number leaving, which (ignoring terms of order dt^2) is given by:

$$(G(R+dt)- G(R)) + G(R+dt)\lambda dt(1-H(R+dt)))$$

The first term is the employed workers whose tenure increases beyond $R+dt$, and the second is those who accept offers of higher tenure. Equating these two flows gives:

$$G(R+dt) - G(R) = \mu dt - G(R+dt)dt(\mu + \lambda(1 - H(R+dt))$$

Letting $dt \rightarrow 0$ gives the result. ∎

PROOF of PROPOSITION 4: $\prod$ is continuous and right-differentiable, and:

$$\Pi'(R) = (Y(R) - s)G'(R) + Y'(R)G(R) \tag{A3}$$

At mass points of H, G' and Y', and hence Π', are discontinuous. For small ε:

$$\begin{aligned}\Pi'(R+\varepsilon) - \Pi'(R) = \left[2\mu Y'(R) - (\mu + \lambda(1 - H(R)))\Pi'(R)\right]\varepsilon \\ + \lambda s\left[H(R) - H(R+\varepsilon)\right]G(R+\varepsilon)\end{aligned} \tag{A4}$$

<u>Step 1</u>: Suppose there is an interval $[R_L, R_H)$ on which H is continuous and strictly increasing. All points in the interval must be optimal for some firm, so $\Pi'(R) = 0 \; \forall R \in [R_L, R_H)$. Then from (A3) $Y'(R) \leq 0 \; \forall R \in [R_L, R_H)$. But from (A4), $\Pi'(R) = 0$ and $Y'(R) \leq 0$ and $H(R) < H(R+\varepsilon) \Rightarrow \Pi'(R+\varepsilon) < 0$ - a contradiction, so there is no such interval.

<u>Step 2</u>: Now suppose there is more than one mass point. Let R_L and R_H be the two mass points at lowest values of R. On $[R_L, R_H)$, H is constant so Y' and Π' are differentiable:

$$Y''(R) = (\mu + \lambda(1 - H(R_L)))Y'(R) \tag{A5}$$

$$\Pi''(R) = 2\mu Y'(R) - (\mu + \lambda(1 - H(R_L)))\Pi'(R) \tag{A6}$$

For R_L and R_H to be optimal choices $\Pi'(R_L) \leq 0$ and $\Pi'(R_H -) \geq 0$.

In an equilibrium with positive profits, $Y(R_L)>s$, $G(R_L)=\mu/(\mu+\lambda)$ and $H(R_L)>0$ so Lemma 5 $\Rightarrow G'(R_L) > 0$. Hence $Y'(R_L)<0$, and so $Y'(R) < 0 \; \forall R \in [R_L, R_H)$. Then from (A6) if

$\Pi'(R_0) = 0$ for any R_0 in $[R_L, R_H)$, $\Pi'(R) < 0\ \forall R \in [R_0, R_H)$. Hence $\Pi'(R_H -) < 0$ which is a contradiction, and the proof of (i) is complete.

<u>Step 3</u>: For (ii), suppose there is a single mass point R_0 of mass 1. To demonstrate the necessary and sufficient conditions for this to be an equilibrium is algebraically messy but straightforward. From (9):

$$Y' = (\mu + \lambda)Y - (p - c)\ \forall R < R_0; \qquad Y' = \mu Y - (p - c)\ \forall R_0 \le R < 0;\ \text{and } Y(0)=0.$$

From Lemma 5:

$$G(R)=\mu/(\mu+\lambda)\ \forall\ R_u \le R < R_0;\ G' = \mu(1 - G)\ \forall R_0 \le R < 0;\ \text{and } G(R_0)=\mu/(\mu+\lambda).$$

This system can be solved explicitly for Y and G, to obtain $\Pi(R)$.

Then, R_0 is an equilibrium if and only if the following conditions hold[15]:

(a) $R_0 \ge R_u$; (b) $\Pi(R_0) \ge 0$;(c) $\Pi'(R_0) \le 0$; and (d) $\Pi'(R_0 -) \ge 0$ or $R_0 = R_u$.

From the solution for Π we obtain $\Pi'(R_0) \le 0 \Leftrightarrow e^{\mu R_0} \ge \dfrac{\lambda(p - c - \mu s)}{(\mu + \lambda)(p - c)}$

and $\Pi'(R_0 -) \ge 0 \Leftrightarrow e^{\mu R_0} \ge \dfrac{\lambda}{\mu + \lambda}$. Also, solving (5) for $\tilde{V}$ gives: $\dfrac{\mu V_0 - c}{p - c} = e^{\mu R_0}$.

Combining this with the four conditions above gives the conditions in (ii).

<u>Step 4</u>: For (iii), consider R_L and R_H again. In an equilibrium with zero profits, $Y(R_L)=s$ and $Y(R_H)=s$, $Y \le s\ \ \forall R \in [R_L, R_H)$. But then $Y = s$ and $Y' = 0\ \forall R \in [R_L, R_H)$ since otherwise $Y' < 0$ and $Y \le s$ somewhere, and then from (A5) $Y<s$ at R_H. Substituting in (9), gives $1 - H(R_L) = \alpha$. Then, from (9) $Y' < 0\ \ \forall\ R \ge R_H$ so there can be no other mass points above R_H. But also, since $Y=s$ at $\bar{R}$, $R_H = \bar{R}$, so $V_0 = \tilde{V}(0; -R_H) = p/\mu - s$. It can then be verified as in the proof of (ii) that $\Pi'(R_L) = 0$ and $\Pi'(R_L -) \ge 0$ for any $R_L \in [R_u, R_H)$, and the proof of (iii) is complete. ∎

PROOF of **PROPOSITION 5**: Let C_0 be a contract with wage profile $w(T)$ satisfying the three conditions, with value profile $V(T)$ to the worker, and $V_0 = V(0)$. Since it is possible for the worker to stay in the match until he retires, $V(T) \ge W(T)\ \forall T$.

<u>Step1</u>: Using (10) and (5) :

$$V'(T) - W'(T) = \mu(V(T) - W(T)) - \lambda \max[V_0 - V, 0] \tag{A7}$$

So $V'(0) \ge W'(0) \ge U'(0) > 0$ and hence $V(T) > V_0$ in an interval $(0, T_0)$.

From (A7): $V(T_0) - W(T_0) \ge V_0 - W(0)$ so $V(T_0) > V_0$ and hence $V(T) > V_0$ for all $T>0$.

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[15] It can be verified that second order conditions are satisfied if these conditions hold.

So C_0 is accepted by unemployed workers only, and they never quit. Hence $V(T)=W(T)\ \forall T$, and V_0= W(0)=$U(0)$. From (1) $V_u = \dfrac{b+\lambda V_0}{\lambda+\mu}$, and $b/\mu \le V_u \le V_0$. The firm's expected payoff from C_0 is:

$$\Pi_0 = \left(\frac{p}{\mu} - s - V_0\right)\frac{\mu}{\lambda+\mu} = \frac{p-\mu s-\mu V_0}{\lambda+\mu}$$

Step 2: Let $G(T)$ be the steady-state measure of employed workers with tenure less than or equal to T, plus the unemployed, who effectively have tenure zero. $G(T)$ satisfies[16]

$G'(T) = \mu(1-G(T))$ and $G(0)=\mu/(\mu+\lambda)$. Solving: $G(T) = 1 - \dfrac{\lambda}{\lambda+\mu}e^{-\mu T}$. A worker with a contract C_0 and tenure $T>0$ has $V(T) > U(T)$. U is a continuous and strictly increasing function of tenure, so the steady-state distribution of workers' contract valuations, $P(V)$, satisfies $P(V) < G(U^{-1}(V))$.

Inverting U gives: $$\exp(-\mu U^{-1}(V)) = 1 - \frac{\mu(\mu V-b)}{\lambda(p-\mu s-\mu V)}$$

and hence: $$P(V) < \frac{\mu}{\lambda+\mu}\left(\frac{p-\mu s-b}{p-\mu s-\mu V}\right)$$

If firm offers a contract of higher initial value $V>U(0)$, the expected payoff is:

$$\Pi_1 = \left(\frac{p}{\mu} - s - V\right)P(V) < \Pi_0$$

Step 3: We already know from Proposition 4 that no firm can make higher profits by offering a contract of lower initial value. Finally, no other contract of initial value $U(0)$ can give higher payoff than C_0, since it keeps the worker for ever. So it is a Nash equilibrium for all firms to offer C_0. ▯

PROOF of **PROPOSITION 6**: Step 1: In a full-turnover equilibrium, a 2–firm offers a contract worth $J^*(p_1)$ and attracts both unemployed workers and workers already in 1–matches; and a 1–firm offers a contract worth V_u.

So from equation (1) $J^*(p_i) = p_i/\mu$ for $i = 1,2$, and

$$V_u = \frac{\mu b + \frac{1}{2}\lambda p_1}{\mu(\mu+\frac{1}{2}\lambda)}$$

[16] The differential equation is derived in the same way as in Lemma 5.

The steady-state condition for e_1, the measure of employed workers in 1-matches, is:

$e_1(\mu+\frac{1}{2}\lambda)=\frac{1}{2}\lambda u$ where $u=\mu/(\mu+\lambda)$.

Step 2: The condition for this equilibrium to exist is that p_2–firms prefer to offer contracts worth $J^*(p_1)$ rather than V_u.

Contracts worth $J^*(p_1)$ give payoff: $\left(J^*(p_2)-J^*(p_1)\right)(u+e_1)=\dfrac{p_2-p_1}{\mu+\frac{1}{2}\lambda}$

whereas contracts worth V_u give: $\left(J^*(p_2)-V_u\right)u=\dfrac{\mu(p_2-b)+\frac{1}{2}\lambda(p_2-p_1)}{(\lambda+\mu)(\mu+\frac{1}{2}\lambda)}$

Comparing these gives condition (13).

Step 3: In a zero-turnover equilibrium, both p_1–firms and p_2–firms offer a contract worth V_u that attracts unemployed workers only. As before equation (1) gives $J^*(p_i)=p_i/\mu$ for i=1,2, but now $V_u=b/\mu$.

The steady-condition for e_1 is $\mu e_1=\frac{1}{2}\lambda u$.

Step 4: This equilibrium exists if 2–firms prefer to offer contracts worth V_u rather than contracts worth $J^*(p_1)$.

Contracts worth $J^*(p_1)$ give payoff: $\left(J^*(p_2)-J^*(p_1)\right)(u+e_1)=\dfrac{(p_2-p_1)(\mu+\frac{1}{2}\lambda)}{\mu(\mu+\lambda)}$

whereas contracts worth V_u give: $\left(J^*(p_2)-V_u\right)u=\dfrac{p_2-b}{\lambda+\mu}$

Comparing these gives the reverse of inequality (13). $\square$

PROOF of **PROPOSITION 7**[17]: Look for an equilibrium in which:

- all 1-firms offer T=0 in $S(V_{01},p_1,b)$ in which the wage rises at $\overline{T}_1$;
- a fraction f of 2-firms offer T_2 in $S(V_{01},p_2,b)$ in which the wage rises at $\overline{T}_2$;
- the remaining 2-firms offer T_L in $S(V_{01},p_2,b)$;

where $0\le T_L<\overline{T}_1\le T_2<\overline{T}_2$. It is convenient to write $k\equiv\dfrac{\lambda q}{\mu}$ and:

$x\equiv\exp(-(1+k)\mu T_L);\ y\equiv\exp(-(1+kf)\mu(T_2-T_L));\ z\equiv\exp(-(1+kf)\mu(\overline{T}_1-T_L))$.

Step1: The steady-state distribution of tenure

Let $G_i(T)$ be the measure of workers in i-firms with tenure $\le T$, and n_1 be total employment in 1-firms. Then $G_1(0)=0$, $G_2(T_L)=0$, and:

[17] An outline proof is given here; full algebraic details are available on request.

For $0 \le T \le T_L$: $\quad G_1' = \mu\left(ku - (1+k)G_1(T)\right)$

and for $T_L \le T$: $\quad G_1' = \mu\left(ku - k(1-f)G_1(T_L) - (1+fk)G_1(T)\right)$

For $T_L \le T \le T_2$: $\quad G_2' = \mu\left(k(1-f)(u+G_1(T_L)) - (1+fk)G_2(T)\right)$

And for $T_2 < T$: $\quad G_2' = \mu\left(k(1-f)(u+G_1(T_L)) + kf(u+n_1) - G_2(T)\right)$

This system can be solved to obtain:

$G_1(T_L) = \dfrac{ku}{1+k}(1-x)$ and $n_1 = \dfrac{ku}{1+k}\left(1 + \dfrac{k(1-f)}{1+fk}x\right)$

The measure of workers who accept contracts with $T=T_L$ is $u + G_1(T_L) = \dfrac{1-kux}{1+k}$

$G_2(\overline{T}_1) = \dfrac{k(1-f)(1-kux)}{(1+k)(1+fk)}(1-z)$; $G_2(T_2) = \dfrac{k(1-f)(1-kux)}{(1+k)(1+fk)}(1-y)$.

<u>Step 2</u>: <u>Value functions</u>

Let $X_i(T)$ and $V_i(T)$ be the payoffs for firm i and its workers at tenure T. Let $r_i = p_i - c$. Then:

$T<0$: $\quad X_i'(T) = (\mu+\lambda)X_i - r_i$

$0 \le T < T_L$: $\quad X_i'(T) = (\mu + \frac{1}{2}\lambda)X_{i1} - r_i$

For firm 1: $\quad T_L \le T < \overline{T}_1$: $\quad X_1'(T) = (\mu + \frac{1}{2}\lambda f)X_1 - r_1$ and $X_1(\overline{T}_1) = 0$

For firm 2: $\quad T_L \le T < T_2$: $\quad X_2'(T) = (\mu + \frac{1}{2}\lambda f)X_2 - r_2$

$\quad T_2 \le T < \overline{T}_2 \quad X_2'(T) = \mu X_2 - r_2 \quad$ and $X_2(\overline{T}_2) = 0$

Also, the two contracts have the same value for the worker for $0 \le T \le \overline{T}_1$, and so $V_2(\overline{T}_1) = V_1(\overline{T}_1) = \dfrac{p_1 + \frac{1}{2}\lambda f V_2(T_2)}{\mu + \frac{1}{2}\lambda f}$. Combining this condition with the differential equation linking $V_2(\overline{T}_1)$ and $V_2(T_2)$, which is $V_2'(T) = (\mu + \frac{1}{2}\lambda f)V_2 - c - \frac{1}{2}\lambda f V_2(T_2)$, we can solve for $V_2(T_2)$ and hence obtain a more useful boundary condition for X_2:

$\mu X_2(T_2) = r_2 - \dfrac{z}{y}r_1$. Solving this system for X_1 and X_2 gives:

$\mu X_1(\overline{T}_1) = 0$; $\quad \mu X_1(T_L) = \dfrac{r_1(1-z)}{1+fk}$; $\quad \mu X_1(0) = r_1\left(\dfrac{1-x}{1+k} + \dfrac{x(1-z)}{1+fk}\right)$ and

$\mu X_2(T_L) = \dfrac{r_2(1-y)}{1+fk} + r_2 y - r_1 z$; $\quad \mu X_2(\overline{T}_1) = \dfrac{r_2(1+\frac{y}{z}fk)}{1+fk} - r_1$; $\quad \mu X_2(0) = \dfrac{r_2(1+kxy)}{1+k} - r_1 xz$.

Step3: Payoff functions

For both firms, payoff per offer is $\Pi_i(T) = X_i(T)(u + G_1(T) + G_2(T))$ for $0 \le T \le \overline{T}_1$.

Firm 2 can also offer $T > \overline{T}_1$ and get $\Pi_2(T) = X_2(T)(u + n_1 + G_2(T))$

Using the equations for G and X above, it can be verified that:

$\Pi_i' = \mu X_i(T) - r_i(u + G_1(T) + G_2(T))$ for $0 \le T \le \overline{T}_1$ and

$\Pi_2' = \mu X_2(T) - r_2(u + n_1 + G_2(T))$ for $0 \le T \le \overline{T}_1$

Step 4: Equilibrium Conditions

Inspection of the payoff functions leads to the following condition:

E1: T=0 is optimal for firm1 if and only if $\mu X_1(0) = r_1 u$. Substituting for $X_1(0)$ from above, this is equivalent to: $ku(1+fk) + x(1-f)k - xz(1+k) = 0$

Similarly:

E2: T_L is the optimal choice for firm 2 in the range $T < \overline{T}_1$ if and only if:

$$\frac{r_2(1-y)}{1+fk} + r_2 y - r_1 z = r_2 \frac{1-kux}{1+k}$$

E3: $T_2 = \overline{T}_1$ is the optimal choice for firm 2 in the range $T \ge \overline{T}_1$ if and only if z=y and:

$$r_1 \ge \frac{r_2}{1+fk}\left(fk + \frac{zk(1-kux)(1-f)}{1+k} \right)$$

E4: $T_2 > \overline{T}_1$ is the optimal choice for firm 2 in the range $T \ge \overline{T}_1$ if and only if z>y and:

$$\frac{z}{y} r_1 = \frac{r_2}{1+fk}\left(fk + \frac{yk(1-kux)(1-f)}{1+k} \right)$$

Step 5: Analysis of Equilibria

Zero-Turnover: In a zero-turnover equilibria f=0 and T_L=0, so x=1. It can be verified E1 and E2 cannot both be satisfied if $r_2 > r_1$. Hence there is no such equilibrium.

Full-Turnover In this case f=1. O1 reduces to xz=ku, and this determines $\overline{T}_1$. Then either

E3: $r_1 \ge \dfrac{r_2 k}{1+k}$ or E4: $\dfrac{z}{y} r_1 = \dfrac{kr_2}{1+k}$ must hold.

(i) So, if $\dfrac{r_2}{r_1} > \dfrac{1+k}{k}$, E3 determines z/y and hence T_2, which is optimal for $T \ge \overline{T}_1$. It can be verified that E2 has no solution for $T_L < \overline{T}_1$, so T_2 is the global optimum for firm 2 and we have an equilibrium.

(ii) Otherwise, if $\frac{1+k}{1+k^2u^2} < \frac{r_2}{r_1} \le \frac{1+k}{k}$, $\overline{T}_1$ is the optimum in $T \ge \overline{T}_1$ and is the global optimum since again E2 has no solution for $T_L < \overline{T}_1$.

(iii) If $\frac{r_2}{r_1} \le \frac{1+k}{1+k^2u^2}$, $\overline{T}_1$ is the optimum in $T \ge \overline{T}_1$ and gives payoff $\mu\Pi_2(\overline{T}_1) = r_2 - r_1$.

O2 has a solution for x that determines T_L and gives payoff:

$\mu\Pi_2(T_L) = r_2\left(\frac{1-kux}{1+k}\right)^2$. Then for equilibrium we require $\Pi_2(\overline{T}_1) \ge \Pi_2(T_L)$,

which, after a little manipulation gives:

$\frac{r_2}{r_1} \ge K \equiv \frac{1+k}{A^2+k}$ where $A \equiv \frac{2ku}{1+u^2k^2} = \frac{4\lambda(\mu+\lambda)}{\lambda^2+4(\mu+\lambda)^2} \in (0,1)$

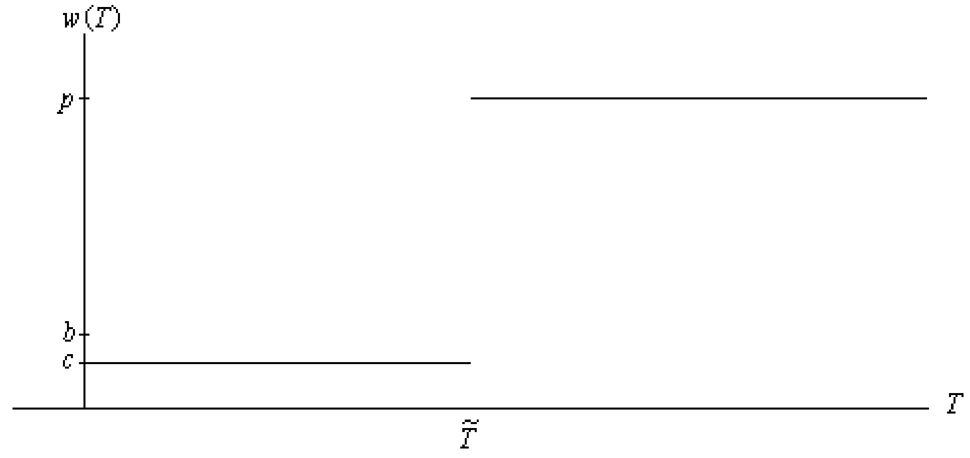
<u>Partial-Turnover</u> Suppose $\frac{r_2}{r_1} < K$, and look for an equilibrium with f<1, and y=z. Then:

(i) E1 and E2 have a solution for x and z that determines T_L and $\overline{T}_1$ such that $0 < T_L < \overline{T}_1$.

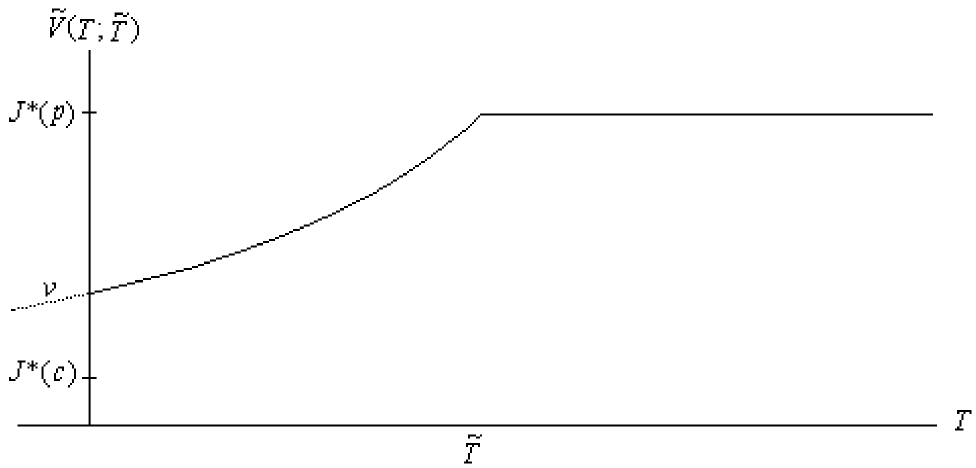
(ii) E3 holds, so $\overline{T}_1$ is the optimum for firm 2 in $T \ge \overline{T}_1$.

(iii) $\mu\Pi_2(T_L) = r_2\left(\frac{1-kux}{1+k}\right)^2$ and $\mu\Pi_2(\overline{T}_1) = (r_2 - r_1)(u + n_1 + G_2(\overline{T}_1))$. Let $S(f)$ represent the difference between the payoffs as f varies: $S(f)=\Pi_2(\overline{T}_1) - \Pi_2(T_L)$. S is continuous in f. We know from 2. above that $S(1)$<0. Then either $S(0)$<0, in which case there is an equilibrium with f=0, or $S(0)$>0, in which case there is some value of f such that S=0, and this gives an equilibrium. □

Figure 1: The step-contract *S*(*v*,*p*,*c*)

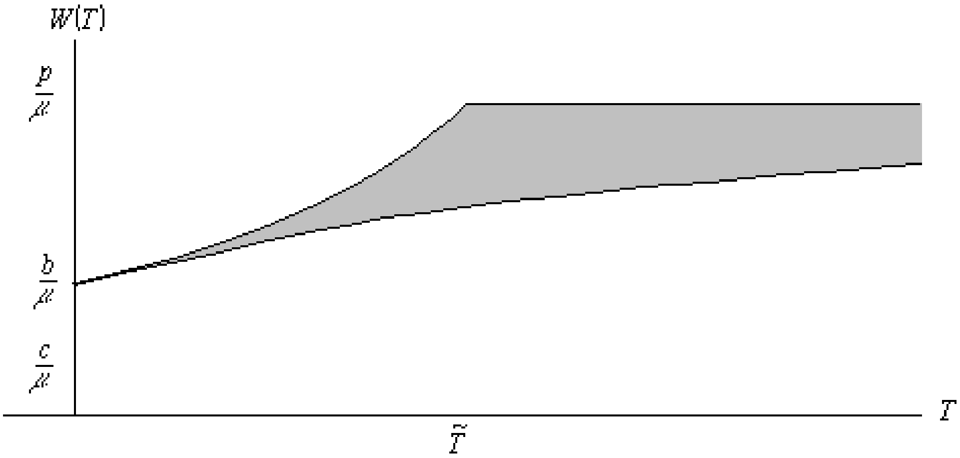


(a) Wage Profile

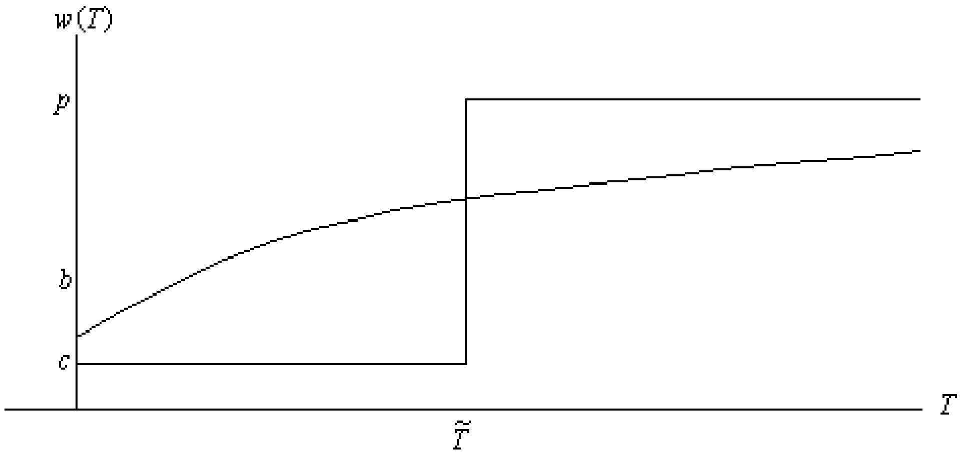


(b) Value Profile

Figure 2: Equilibrium Contract Profiles



(a) Region of equilibrium value profiles



(b) Wage profiles corresponding to boundaries of equilibrium region

Figure 3: Equilibrium Turnover with Two Types of Firms