DEMAND IS HETEROGENOUS IN GRANDMONT'S MODEL

By John K.-H. Quah

Abstract: We show that Grandmont's (1992) model of demand heterogeneity can be a model of heterogeneity in the complementary or sign-balancing sense. By this we mean that heterogeneity has the following form: given a change in price, agents respond heterogenously - some by increasing their expenditure share on a good, others by diminishing it, so that the average expenditure share of all goods remain approximately unchanged.

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THIS NOTE IS A COMMENT on the nature of the behavioral heterogeneity in Grandmont's (1992) model of market demand. A discussion of this issue appears to have arisen following K. Hildenbrand's (1998) paper, "On Grandmont's Modelling of Behavioral Heterogeneity." In that paper, K. Hildenbrand argues that in Grandmont's model, "due to the specific parametrization, increasing dispersedness of the parameters leads to an increasing concentration of the demand functions", so that Grandmont ends up modelling "increasing similarity". This paper shows that while K. Hildenbrand has highlighted some interesting features of the behavioral heterogeneity model employed by Grandmont and other authors following him (including Quah (1997)), the claim he makes is not true, or at least not always true, and Grandmont's model is in a very reasonable sense a model of demand heterogeneity.

I begin with a simple account of a Grandmont-type argument. Following my paper (Quah (1997)), I will not consider affine transformations (used in Gradmont (1992)) but consider instead the smaller set of homothetic transformations. This is slightly simpler than Grandmont's (1992) setup, and we can see here, with greater clarity, both the working of the heterogeneity argument and also the nature of the possible objections to it.

Given a function $f : R^n_{++} \times R_{++} \to R^n_{++}$ we can define another function f_t by $f_t(p, w) = e^t f(p, we^{-t})$, where t is a real number. We call the collection $\{f_t\}_{t \in R}$ the homothetic class of f, and refer to f as the *generator*.

This is just one of many possible transformations of the function f. What makes homoethetic transformations special is its preservation of the possible rationality properties on f. It is straightforward to check that if f satisfies the weak axiom, then so will its homothetic transformation f_t; and if f is generated by the utility function u, then its homothetic transformation f_t is generated by the utility function $u_t(x) = u(e^{-t}x)$. These characteristics allow the modeller to use homethetic transformations (and more generally affine transformations) to model a market where agents have different demand functions in a simple parametric fashion, secure in the knowledge that individual demands are at least compatible with individual rationality.

To see how the heterogeneity argument works, we consider a market consisting of agents facing the same prices and with the same income level but having different demand functions. Let P in R^l_{++} be a compact set of prices, and let W be compact interval in R_{++}, representing possible income levels. Let $f : R^l_{++} \times R_+ \to R^l_{++}$ be some function (possibly a demand function generated by a utility function). *Crucially, we assume that each agent in this market has a demand function which coincides with f_t for some t in the set $P \times W$.* This agent's demand may or may not coincide with f_t outside the set $P \times W$; we do not really care, for what we are interested in is the behavior of market demand in the set $P \times W$. (This distinction turns out to be quite important - more on this later.) *Market demand* is

$$F(p, w) = \int_R f_t(p, w) h(t) dt$$

where h is a density function defined on R. For each f_t we can also define s_t by $s^i_t(p, w) = p^i f^i_t(p, w)/w$ for $i = 1, 2, ..., l$; in other words s^i_t is the share of expenditure going to good i when demand is f_t. Similarly, we write the average share of expenditure going to good i in this market as $S^i(p, w)$; obviously,

$$S^i(p, w) = \frac{p^i F^i(p, w)}{w} = \int_R s_t(p, w) h(t) dt.$$

The central claim of the heterogeneity argument relates the behavior of S (or F) in the set $P \times W$ with the density function h. In particular when h is sufficiently flat (in a sense I will make precise), S becomes largely independent of income and aggregate behavior takes on a homothetic character. For the sake of completeness, the argument, which follows Grandmont (1992) and Quah (1997), is repeated here.

PROPOSITION: *Suppose that $s^i(p,w) < M^i$, for all p in P and $w > 0$, and $W = [w_*, w^*]$. Then, for any (p,w) and $(p,\bar{w})$ in $P \times W$,*

$$|S^i(p,w) - S^i(p,\bar{w})| \leq M^i \ln\left(\frac{w^*}{w_*}\right) \int_R |h'(t)|dt.$$

(Note that, by its definition, $s^i(p,w)$ is always less than 1, so M^i always exists.)

Proof: Consider w and $\bar{w}$ in W, with $\bar{w} > w$. By substitution, we can write

$$\begin{aligned}
S^i(p,\bar{w}) &= \int_R s^i(p,\bar{w}e^{-t})h(t)dt \\
&= \int_R s^i(p,we^{\ln(\bar{w}/w)-t})h(t)dt \\
&= \int_R s^i(p,we^{-t})h\left(\ln\left(\frac{\bar{w}}{w}\right)+t\right)dt.
\end{aligned}$$

So then

$$\begin{aligned}
|S^i(p,\bar{w}) - S^i(p,w)| &= \left|\int_R s^i(p,we^{-t})\left[h(\ln(\bar{w}/w)+t) - h(t)\right]dt\right| \\
&\leq M^i \int_R |h(\ln(\bar{w}/w)+t) - h(t)|\,dt \\
&= M^i \int_R \left|\int_0^{\ln(\bar{w}/w)} h'(t+v)dv\right| dt \\
&\leq M^i \int_0^{\ln(\bar{w}/w)} \int_R |h'(t+v)|dtdv \\
&\leq M^i \ln\left(\frac{w^*}{w_*}\right) \int_R |h'(t)|dt.
\end{aligned}$$

. QED

This result says that as $\int_R |h'(t)|dt$ becomes small, for any fixed p in P, $S(p,w)$ becomes increasingly insensitive to w in W. As an example of how h can become increasingly small, consider the density functions $h_n(t) = n^{-1}h(tn^{-1})$ where h is any density function with $\int_R |h'(t)|dt$ finite. Then as n goes to infinity, $\int_R |h_n'(t)|dt$ goes to zero.

Quah (1997) uses variations of this basic Proposition to establish the uniqueness and and stability of the equilibrium price in exchange and production economies. Grandmont (1992) employs affine transformations, and essentially because this is a larger set than homothetic transformations, heterogeneity over these transformations allows him to obtain stronger aggregate properties; in particular, aggregate expenditure shares becomes increasingly independent of prices, i.e., market demand, in essentially the same setting as the one here, acquires approximately Cobb-Douglas, and not just homothetic, properties.

Is this a good model of demand heterogeneity? I will now make, in this simple context, a number of observations which are essentially analogous to the ones made by K. Hildenbrand (1998), and to explain why one could construe difficulties with the model.

Firstly, the word "heterogeneity" suggests difference, so this means that the demand f_t must be different for different ts for one to claim that a market made up of such transformations exhibit heterogeneity. This immediately implies that the generator f cannot be linear in income, because in this case $f_t = f$ and so all the agents in the market will have the same demand in $P \times W$.

But the problem is worse than this. Suppose that for some sequence h_n of density functions, $\int_R |h_n'(t)|dt$ is going to zero; clearly then, on any compact set T of the real line, $\int_T h_n(t)dt$ goes to zero, or to put it more intuitively, if a density function is getting flatter and flatter, then it has to be spreading to the left or to the right. Let K be a compact interval in R_+ (the positive real numbers excluding zero). Define $T = \{t \in R : we^{-t} \in K$ for some $w \in W\}$; since this is compact, $\int_T h_n(t)dt$ goes to zero. Put another way, as h_n becomes flat, the values of t that "predominate" are those that are very small or very big; since $s_t(p,w) = s(p, we^{-t})$, the behavior of s at very low or very high income levels become the most relevant. For example, if $t = -\ln 1000$, then the behavior of $s_{-\ln 1000}$ in $W = [w_*, w^*]$ is determined by the behavior of s in the interval $[1000w_*, 1000w^*]$. If $t = \ln 1000$, then the behavior of $s_{\ln 1000}$ is determined by the behavior of s in the interval $[w_*/1000, w^*/1000]$.

Suppose now that with p kept fixed, for some good i, s^i looks like Case 1, with limits as income goes to zero and infinity. Then it is quite clear that as t becomes very small or very big, s^i_t *itself* becomes increasingly insensitive to w in the interval W, and it is precisely these values of t that predominate as h_n becomes flat. So while our Proposition is still true, and $S^i(p,w) = \int_R s^i_t(p,w)h(t)dw$ becomes increasingly insensitive to w in W, this happens because the market is simply packed more and more with agents that individually display the same behavior. Notice that there *is* heterogeneity, in the sense that agents have different demand functions, but clearly it is misleading in this case to claim that heterogeneity *leads to* homotheticity.

So for heterogeneity that leads to homotheticity, the function s must look like Cases 2 or 3. In Case 2, as income goes to infinity, s^i has no limit, so that s^i_t will not be approximately constant in the interval W however small is t. So if the h_ns are such that $\int_{R_-} h_n(t)dt$ tends to 1, for s^i in Case 2, we *do* have a true model of heterogeneity, in the following sense: the demand functions of the agents are individually very far from being linear in income in the set W; as income is increased a little, say from w to w', individual agents with different ts give a *heterogenous response* to this change in income, and these heterogeneous responses negate each other so that average expenditure share on good i, S^i, stays approximately unchanged between w and w'. One could refer to this type of demand behavior, quite naturally, as *complementary* or *sign-balancing* heterogeneity (the terms are loosely borrowed from Villemeur (2001) and W. Hildenbrand and A. Kneip (1999) respectively).

In Case 3, s becomes increasingly wriggly as income goes to zero, and once again in this case, we have a true model of heterogeneity if $\int_{R_+} h_n(t)dt$ tends to 1.

What these observations show is that this parametric model captures complementary or sign-balancing heterogeneity only if the generator s satisfies certain conditions - in particular, it has to be like Cases 2 or 3, but not Case 1. This is a useful observation to make, but K. Hildenbrand (1998) goes further. He appears to have two closely related objections, which I will try to transpose to this context. Firstly, he objects to the reliance on the marginal behavior of s, i.e., the behavior of s at very low or very high incomes, and secondly, he thinks that the marginal conditions imposed on s are implausible. At least that is what I get from a close reading of his paper. So referring to Grandmont's model he says "to *base* a theory on the speed, i.e., whether demand runs to infinity slower, faster or with equal speed as price runs to zero is not acceptable" (The emphasis is his; in the model described here, for "prices" read "income.") Furthermore, on page 3 of his paper, he describes the behavior in Cases 2 or 3 as "hysteric-like" (the inverted commas are his; once again, he is directly referring to Grandmont's model, but it translates naturally to Cases 2 and 3 here). In this way, having eliminated Cases 2 and 3 as unacceptable, he ends up finding only something like Case 1 acceptable, and so concludes that Grandmont and those "many scientific authors" employing similar methods are in fact modelling "behavioral similarity."

I argue that Cases 2 and 3 are perfectly acceptable, provided the domain of prices and income over which demand is modelled by the transformations (homothetic or affine) is not the whole positive orthant. Note that the behavior in these cases are not incompatible with individual rationality. However, it is true that, firstly, restrictions are indeed placed on the boundary behavior of s, and secondly, the behavior is arguably implausible, in the sense that introspection suggests that an agent will not exhibit demand behavior like that. But is this relevant?

The function s may be implausible, but the demand of agents in the model is not s, but rather s_t as defined *in the set* $P \times W$, for different values of t - the issue is whether these s_t are plausible *in* $P \times W$, and nothing observed here or in K. Hildenbrand (1998) suggests that they are not. We may reasonably require that s_t be plausible in $P \times W$, but while it is nice if s is plausible in $R^l_{++} \times R_{++}$, the latter condition cannot reasonably be deemed essential - s (or, more properly, f) is after all just a generator.

Since a homothetic (or in Grandmont's case, affine) transformation of f will inherit any implausible boundary behavior of f, excluding Cases 2 or 3 may be reasonable if agents are modelled to have demand functions that coincide with these transformations *on the whole positive orthant*, $R^l_{++} \times R_+$. Indeed, this is the way the models in Grandmont (1992) and Quah (1997) are presented. However, a careful reading of those papers will reveal that this is just a convenient simplification. The authors in those papers were principally interested in the global uniqueness and stability of the equilibrium price. For simplicity, consider an exchange economy. Then what goes on is this: firstly, some assumption is made (and in fact there are many variations, beyond the ones supplied by the authors) which guarantee that no equilibrium price exists outside some strictly compact set of prices P; given P, and given the endowments of the agents, we can define W, a compact interval of strictly positive incomes which agents in the economy can achieve assuming that prices are in P. Secondly, in the set $P \times W$ (but not necessarily outside that set), agents have demand functions which belong to homothetic classes in the case of Quah (1997) and affine classes in the case of Grandmont (1992). Increasing heterogeneity and other assumptions then guarantee a nice aggregate structure to aggregate demand in $P \times W$, leading to uniqueness and stability of the equilibrium price.

Before I conclude, and at the risk of muddying the waters a little, I make a few observations on the model of heterogeneity of W. Hildenbrand and A. Kneip (1999). These authors have a model in which agents' individual behavior are allowed to depart from the Cobb-Douglas form, but large departures occur for *different* agents at *different* parts of the price space (hence *heterogeneity*). This has the effect that at any single price vector, only a small fraction of the agents actually have large deviations from Cobb-Douglas behavior and so Cobb-Douglas behavior holds approximately in the aggregate. As an example, the authors consider Grandmont's model in which they impose an additional assumption on the generating function. They assume that the function admits a universal and finite bound on the number of turning points of the expenditure share of good i as a function of the price of good j, for all i and j. (This is Assumption 3 in their paper; in our context, the authors would permit Case 1, but the assumption specifically excludes Cases 2 and 3). They then demonstrate that Grandmont's model, *with this additional assumption*, satisfies heterogeneity in their sense. In view of our discussion of what happens in Case 1, the intuition for their result should be clear. In short, what K. Hildenbrand (1998) calls "increasing similarity" in Grandmont's model is an example of "heterogeneity" in the sense of W. Hildenbrand and A. Kneip (1999).

It is not the objective of this note to judge whether heterogeneity in this sense or in the complementary/sign-balancing sense (as described here, in Cases 2 and 3) is the empirically more significant reason for Cobb-Douglas-like or homothetic-like aggregate behavior. What I hope I have made clear is the following: the Grandmont model is not *necessarily* a model of increasing similarity or of heterogeneity in the sense of W. Hildenbrand and A. Kneip (1999); an alternative interpretation which is just as valid, if not more so, is to view Grandmont's model as a model of complementary or sign-balancing heterogeneity.

References:

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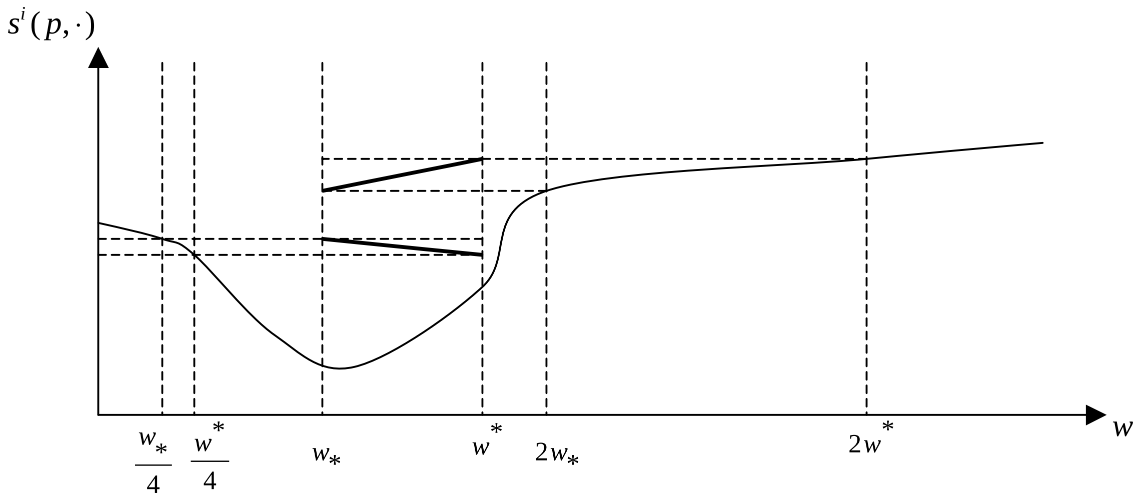
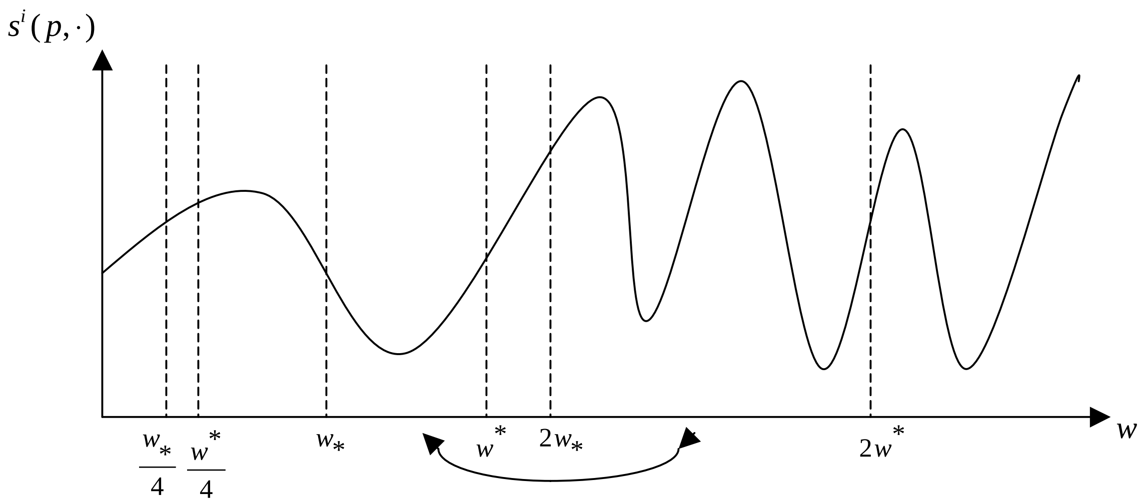
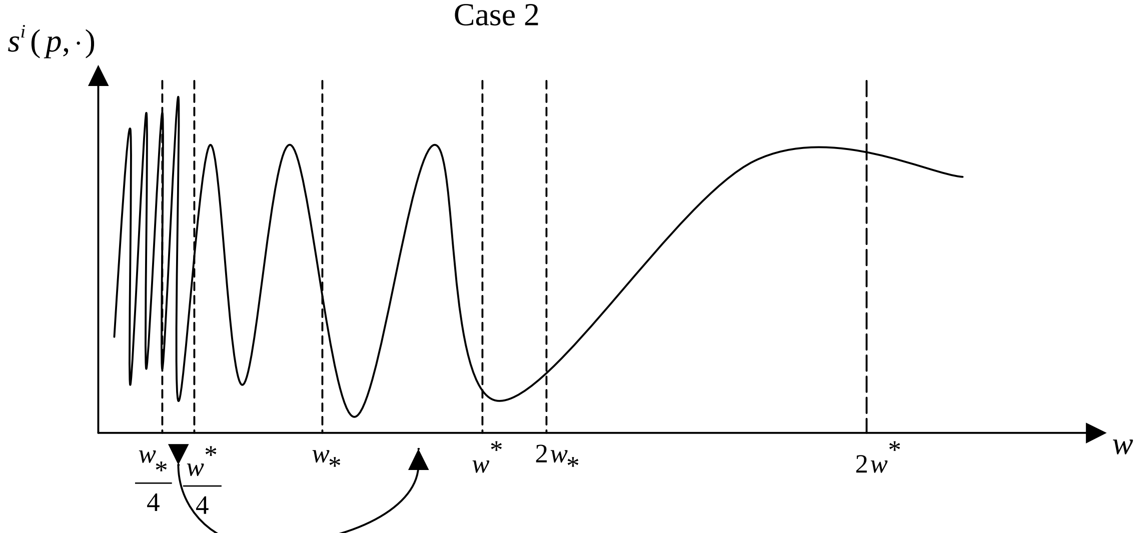


Case 1



Case 2



Case 3