Complex Collective Decisions and the Probability of Collective Inconsistencies

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Abstract. Many groups have to make decisions over multiple interconnected pro-positions. The "doctrinal paradox" or "discursive dilemma" shows that propositionwise majority voting can lead to inconsistent collective outcomes, even when individual judgments are all consistent. How likely is the occurrence of this paradox? This paper develops a simple model for determining the probability of the paradox's occurrence, given various probability distributions over individual judgments. Several convergence results are proved, identifying conditions under which that probability converges to certainty as the number of individuals increases, and conditions under which it vanishes. The model is used for assessing the "truth-tracking" performance of two escape-routes from the paradox, the premise- and conclusion-based procedures. Finally, the present results are compared with existing results on the probability of Condorcet's paradox. It is suggested that the doctrinal paradox is likely to occur under plausible conditions.

A new paradox of aggregation, the "doctrinal paradox" or "discursive dilemma", has been the subject of a growing body of literature in the fields of law, economics and philosophy (Kornhauser and Sager 1986; Kornhauser 1992; Kornhauser and Sager 1993; Chapman 1998; Brennan 2001; Pettit 2001; List and Pettit 2002a, 2002b; Chapman 2001, 2002; Bovens and Rabinowicz 2001a, 2001b). A simple example illustrates the problem. Suppose that a panel of three judges has to decide on whether a defendant is liable under a charge of breach of contract. Legal doctrine requires that the court should find that the defendant is liable (proposition R) if and only if it finds, first, that the defendant did some action X (proposition P), and, second, that the defendant had a contractual obligation not to do action X (proposition Q). Thus legal doctrine stipulates the connection rule ($R \leftrightarrow (P \wedge Q)$). Suppose the opinions of the three judges are as in table 1.

Table 1: The Doctrinal Paradox (Conjunctive Version)

	P	Q	$(R \leftrightarrow (P \wedge Q))$	R
Judge 1	Yes	Yes	Yes	Yes
Judge 2	Yes	No	Yes	No
Judge 3	No	Yes	Yes	No
Majority	Yes	Yes	Yes	No

All judges accept the connection rule, ($R \leftrightarrow (P \wedge Q)$). Further, judge 1 accepts both P and Q and, by implication, R. Judges 2 and 3 each accept only one of P or Q and, by

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implication, they both reject R. If the court applies majority voting on each proposition (including the connection rule), it faces a paradoxical outcome. A majority accepts P, a majority accepts Q, a majority (unanimity) accepts $(R \leftrightarrow (P \wedge Q))$, and yet a majority rejects R. Propositionwise majority voting thus yields an inconsistent collective set of judgments, namely the set $\{P, Q, (R \leftrightarrow (P \wedge Q)), \neg R\}$ (corresponding to the last row of table 1). This set is inconsistent in the standard sense of propositional logic: there exists no assignment of truth-values to the propositions P, Q and R that will make all the propositions in the set simultaneously true. And this outcome occurs even though the sets of judgments of individual judges (corresponding to the first three rows of table 1) are all consistent. The doctrinal paradox is also related to Anscombe's paradox, or Ostrogorski's paradox (Anscombe 1976; Kelly 1989; Brams, Kilgour and Zwicker 1997). Like the doctrinal paradox, these paradoxes are concerned with aggregation over multiple propositions. Unlike the doctrinal paradox, however, they do not explicitly incorporate logical connections between the relevant multiple propositions.

Pettit (2001) has argued that the doctrinal paradox occurs not only in the context of aggregation of judgments according to legal doctrine, but that it poses a more general "discursive dilemma", which any group may face when it seeks to form collective judgments on the basis of reasons. Further, List and Pettit (2002a, 2002b) have shown that the paradox illustrates a general impossibility theorem. According to the theorem, there exists no procedure for aggregating individual sets of judgments over multiple interconnected propositions into collective ones, where the procedure satisfies a set of minimal conditions. The paradox is related to the impossibility theorem in a way that is analogous to the way in which Condorcet's paradox of cyclical preferences is related to Arrow's impossibility theorem.

Versions of the present aggregation problem may arise, for example, when a committee has to make a decision that involves the resolution of several premises; or when a political party or interest group seeks to come up with an entire policy package, where such a package consists of several interconnected propositions. Although the label "doctrinal paradox" will be used here, the more general nature of the problem should be kept in mind.

How serious is the threat posed by this paradox? It is one thing to recognize that a given paradox of aggregation is logically possible. It is another to claim that the paradox is also of practical significance. There are at least two possible reasons why a particular paradox might not (seem to) occur in practice. One is that many of those decision procedures that are used in practice do not *explicitly reveal* the paradox, even when individual views have exactly the pattern that *would* give rise to the paradox. Two such procedures for making decisions over multiple interconnected propositions will be discussed below, namely the so-called premise-based and conclusion-based procedures. These procedures will not produce inconsistent collective sets of judgments, even when a pattern of individual views similar to the one in table 1 occurs. But it will become evident below that the question of how frequently such patterns occur is nonetheless relevant for assessing the performance of the two procedures. A second possible reason why the paradox might not occur in practice is that the patterns of individual views that generate the paradox might themselves be rare.

So how likely is the occurrence of this paradox, or more precisely, how likely is the occurrence of patterns of individual views that generate the paradox? The aim of this paper is to give a theoretical answer to this question. Inevitably, a large range of other important questions raised by the doctrinal paradox cannot be addressed here. In section 1, necessary and sufficient conditions for the occurrence of the paradox are identified. In section 2, a probability-theoretic model is developed for determining the probability of its occurrence, given various assumptions about the probability that individuals hold different sets of judgments. Some convergence results are proved, identifying conditions under which the probability of the paradox's occurrence converges to 1 as the number of individuals increases, and conditions under which that probability converges to 0. In section 3, two escape-routes from the paradox, the premise- and conclusion-based procedures of decision-making, are discussed, and, following two recent papers by Bovens and Rabinowicz (2001a, 2001b), their performance in terms of "tracking the truth" is investigated. The present model yields alternative proofs as well as extensions of some of the results by Bovens and Rabinowicz. It is also shown that, under certain conditions, if each individual is better than random at tracking the "truth" on each of the premises, but not very good at it, then the probability of the occurrence of the doctrinal paradox (and the probability of a discrepancy between the premise- and conclusion-based procedures) converges to 1 as the number of individuals increases. Section 4 addresses extensions and generalizations of the present results. And, in section 5, finally, the present results are briefly compared with existing results on the probability of cycles in the realm of preference aggregation.

Before embarking on the analysis, it is helpful to address one objection. Many of the theoretical results of this paper are convergence results concerning the behaviour of certain probabilities as the number of individuals increases. From the perspective of the typical size of those decision-making bodies faced with problems of aggregation over multiple interconnected propositions, it is not immediately clear why such convergence results, or any results about large numbers of individuals, should be relevant. Typical examples of such decision-making bodies are courts, committees, panels of experts, or parliaments, with between a handful and a few hundred members.

In response to this objection, four points should be noted. First, the theoretical framework developed in this paper is suitable for calculating the relevant probabilities for finite (and indeed small) numbers of individuals too; see in particular appendices 1 and 2. Second, ever since Condorcet's famous work on jury decisions – which provides the motivation for the discussion of "truth-tracking" in sections 3 and 4 –, the convergence behaviour of the probabilities of various voting outcomes has been a central focus of attention, and hence it is theoretically interesting to address Condorcet's traditional questions in the new context of aggregation over multiple interconnected propositions. Third, as table 3 in section 2 illustrates, convergence results may be relevant even to situations of just a few dozen or a few hundred decision-makers, as the speed of convergence of the relevant probabilities may often be quite high. Finally, the results may shed some light on several questions in democratic theory, such as (i) whether it is desirable to introduce large-scale political participation on complex issues by running more referenda over multiple propositions and (ii) what the optimal size of decision-making bodies is for complex collective decisions. In section 5, some relevant anecdotal evidence from referenda in California will be cited.

1 Necessary and Sufficient Conditions for the Occurrence of the Paradox

Suppose that there are n individuals and three propositions, P, Q and R. Suppose further that all individuals accept the connection rule ($R \leftrightarrow (P \wedge Q)$). Admitting only consistent individual sets of judgments over P, Q and R, there are 4 logically possible sets of judgments an individual might hold, as shown in table 2.

Table 2: All logically possible consistent sets of judgments over P, Q and R, given ($R \leftrightarrow (P \wedge Q)$)

Label	Judgment on P	Judgment on Q	Judgment on R
PQ	Yes	Yes	Yes
$P\neg Q$	Yes	No	No
$\neg PQ$	No	Yes	No
$\neg P\neg Q$	No	No	No

Let n_{PQ}, $n_{P\neg Q}$, $n_{\neg PQ}$, $n_{\neg P\neg Q}$ be the numbers of individuals holding the sets of judgments PQ, $P\neg Q$, $\neg PQ$, $\neg P\neg Q$, respectively. A collective inconsistency (a "doctrinal paradox") occurs if and only if there are majorities for each of P and Q, and there is a majority against R. If there are ties, we allow that these may be broken in whichever way collective consistency requires.

Proposition 1. *Given the connection rule* $(R \leftrightarrow (P \wedge Q))$, *there will be a collective inconsistency under propositionwise majority voting if and only if* $(n_{PQ} + n_{P\neg Q} > n/2)$ *and* $(n_{PQ} + n_{\neg PQ} > n/2)$ *and* $(n_{PQ} < n/2)$.

Given unanimous acceptance of $(R \leftrightarrow (P \wedge Q))$, the conditions of proposition 1 are necessary and sufficient for the majority acceptance of the (inconsistent) set of propositions $\{P, Q, (R \leftrightarrow (P \wedge Q)), \neg R\}$.

I make no claims as to whether it is *empirically* plausible to assume that individuals hold consistent sets of judgments. For the present purposes, it is sufficient to note that admitting only consisting individual sets of judgments makes the occurrence of inconsistent collective sets of judgments less likely rather than more likely. If we can still show that, in a large class of cases, collective inconsistencies will occur, then the argument will effectively have been strengthened rather than weakened by the exclusion of inconsistent individual sets of judgments.

2 A Probability-Theoretic Framework

To study the likelihood of the occurrence of collective inconsistencies, we assume that (i) each individual has probabilities p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$ of holding the sets of judgments PQ, $P\neg Q$, $\neg PQ$, $\neg P\neg Q$, respectively (where $p_{PQ} + p_{P\neg Q} + p_{\neg PQ} + p_{\neg P\neg Q} = 1$); and (ii) the judgments of different individuals are independent from each other.

The simplifications implicit in these assumptions follow the classical Condorcet jury theorem. Specifically, we assume (i) identical probabilities for different individuals, and (ii) independence between different individuals. However, it is known in the literature on the Condorcet jury theorem that the types of convergence mechanisms based on the law of large numbers invoked in the present paper apply, with certain modifications, also when probabilities vary across individuals or when there are certain dependencies between individuals (Grofman, Owen and Feld (1983) and Borland (1989)). We turn to a Condorcet jury framework more properly in section 3.

An *impartial culture* is the situation of perfect equiprobability across all logically possible sets of judgments, i.e. $p_{PQ} = p_{P\neg Q} = p_{\neg PQ} = p_{\neg P\neg Q}$. The function for calculating the probability of each logically possible combination of individual sets of judgments for a given number of individuals n is stated in appendix 1.

Moreover, appendix 1 includes a formula for calculating the probability that there will be a collective inconsistency under propositionwise majority voting for various numbers of individuals n and various values of p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$, where the connection rule is $(R \leftrightarrow (P \wedge Q))$. To avoid complications raised by ties, we assume that the number of individuals is odd. Table 3 shows some sample calculations.

Table 3: Probability that there will be a collective inconsistency under propositionwise majority voting (given $(R \leftrightarrow (P \wedge Q))$), for various scenarios

	Scenario 1 $p_{PQ} = 0.25$ $p_{P\neg Q} = 0.25$ $p_{\neg PQ} = 0.25$ $p_{\neg P\neg Q} =$ 0.25	Scenario 2 $p_{PQ} = 0.26$ $p_{P\neg Q} = 0.25$ $p_{\neg PQ} = 0.25$ $p_{\neg P\neg Q} =$ 0.24	Scenario 3 $p_{PQ} = 0.3$ $p_{P\neg Q} = 0.25$ $p_{\neg PQ} = 0.25$ $p_{\neg P\neg Q} =$ 0.2	Scenario 4 $p_{PQ} = 0.24$ $p_{P\neg Q} = 0.27$ $p_{\neg PQ} = 0.25$ $p_{\neg P\neg Q} =$ 0.24	Scenario 5 $p_{PQ} = 0.49$ $p_{P\neg Q} = 0.2$ $p_{\neg PQ} = 0.2$ $p_{\neg P\neg Q} =$ 0.11	Scenario 6 $p_{PQ} = 0.51$ $p_{P\neg Q} = 0.2$ $p_{\neg PQ} = 0.2$ $p_{\neg P\neg Q} =$ 0.09	Scenario 7 $p_{PQ} = 0.55$ $p_{P\neg Q} = 0.2$ $p_{\neg PQ} = 0.2$ $p_{\neg P\neg Q} =$ 0.05	Scenario 8 $p_{PQ} = 0.33$ $p_{P\neg Q} = 0.33$ $p_{\neg PQ} = 0.33$ $p_{\neg P\neg Q} =$ 0.01
$n = 3$	0.0938	0.0975	0.1125	0.0972	0.1176	0.1224	0.1320	0.2156
$n = 11$	0.2157	0.2365	0.3211	0.2144	0.3570	0.3432	0.2990	0.6188
$n = 31$	0.2487	0.2946	0.4979	0.2409	0.5183	0.4420	0.2842	0.9104
$n = 51$	0.2499	0.3101	0.5815	0.2405	0.5525	0.4414	0.2358	0.9757
$n = 71$	≈ 0.2500	0.3216	0.6417	0.2393	0.5663	0.4327	0.1983	0.9930
$n = 101$	≈ 0.2500	0.3362	0.7113	0.2375	0.5798	0.4201	0.1562	0.9989
$n = 201$	≈ 0.2500	0.3742	0.8511	0.2317	0.6118	0.3882	0.0774	≈ 1.0000
$n = 501$	≈ 0.2500	0.4527	0.9754	0.2149	0.6729	0.3271	0.0124	≈ 1.0000
$n = 1001$	≈ 0.2500	0.5426	0.9985	0.1897	0.7366	0.2634	0.0008	≈ 1.0000
$n = 1501$	≈ 0.2500	0.6097	0.9999	0.1676	0.7808	0.2192	0.0001	≈ 1.0000

Slight differences in the probabilities that individuals hold the different possible sets of judgments correspond to substantial differences in the resulting probability that a collective inconsistency will occur under propositionwise majority voting. In the special case of an impartial culture (scenario 1), the probability of the occurrence of a collective inconsistency appears to converge to 0.25 as the number of individuals increases. Slight deviations from an impartial culture, however, entail a completely different convergence pattern. This is confirmed by the following convergence results, proved in appendix 3.

Proposition 2. *Let the connection rule be* $(R \leftrightarrow (P \wedge Q))$.

(a) *Suppose* $(p_{PQ} + p_{P\neg Q} > 1/2)$ *and* $(p_{PQ} + p_{\neg PQ} > 1/2)$ *and* $(p_{PQ} < 1/2)$. *Then the probability of a collective inconsistency under propositionwise majority voting converges to* 1 *as n tends to infinity.*

(b) *Suppose* ($p_{PQ} + p_{P\neg Q} < 1/2$) *or* ($p_{PQ} + p_{\neg PQ} < 1/2$) *or* ($p_{PQ} > 1/2$). *Then the probability of a collective inconsistency* (*given under propositionwise majority voting converges to* 0 *as n tends to infinity*.

Scenarios 2, 3, 5 and 8 in table 3 satisfy the conditions of proposition 2a, and scenarios 4, 6 and 7 satisfy the conditions of proposition 2b. The numerical values in table 3 thus provide illustrations of the convergence mechanisms identified by proposition 2.

The convergence results are effectively a consequence of the law of large numbers. If p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$ are the probabilities that an individual holds the sets of judgments PQ, $P\neg Q$, $\neg PQ$, $\neg P\neg Q$, respectively, then np_{PQ}, $np_{P\neg Q}$, $np_{\neg PQ}$, $np_{\neg P\neg Q}$ are the expected numbers of these sets of judgments across n individuals, and p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$ are the expected frequencies (i.e. the expected numbers divided by n). If n is small, the actual frequencies may differ substantially from the expected ones, but as n increases, the actual frequencies will approximate the expected ones increasingly closely. In particular, if the probabilities p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$ satisfy a set of strict inequalities, the actual frequencies (and by implication the actual numbers) are increasingly likely to satisfy a matching set of strict inequalities. But if these are the inequalities corresponding to the occurrence or absence of a collective inconsistency (compare proposition 1), this means that the probability of the occurrence or absence of such an inconsistency will converge to certainty. The described mechanism will be used to prove other convergence results below. Lemma 1 in appendix 3 captures the mechanism formally.

The results of this section are also useful from the perspective of the general impossibility result on the aggregation of judgments over multiple interconnected propositions. They allow us to determine, under various assumptions about the probability of different individual judgments, how likely it is that a combination (or *profile*) of sets of views across individuals will fall into a problematic domain (one in which collective inconsistencies under propositionwise majority voting occur), and how likely it is that it will fall into an unproblematic one (one in which propositionwise majority voting leads to consistent outcomes).

3 Voting for the Premises Versus Voting for the Conclusion

Premise-based and conclusion-based procedures of decision-making have been proposed as possible escape-routes from the doctrinal paradox (see, for example, Pettit 2001). According to the premise-based procedure, the group applies majority voting on propositions P and Q, the "premises", but not on proposition R, the "conclusion", and lets the connection rule, $(R \leftrightarrow (P \wedge Q))$, dictate the collective judgment on R, effectively ignoring the majority verdict on it. Given the individual judgments in table 1, the premise-based procedure leads to the collective acceptance of P and Q and, by implication, R. According to the conclusion-based procedure, the group applies majority voting only on R, but not on P and Q, thereby effectively ignoring the majority verdicts on these propositions. Given the individual judgments in table 1, the conclusion-based procedure leads to the collective rejection of R. This illustrates that the premise-based and conclusion-based procedures may produce divergent outcomes.

As both Pettit (2001) and Chapman (2002) argue, the premise-based procedure is particularly attractive from the perspective of deliberative democracy, as it prioritises, and "collectivises", the reasons underlying a given overall decision. A key concern of deliberative democracy is to make collective decisions based on publicly defensible reasons. The conclusion-based procedure, on the other hand, focuses solely on the conclusions that individuals privately reach, ignoring these individuals' views on the premises. The conclusion-based procedure thus fails to make the underlying reasons for an overall collective decision explicit at the collective level.

However, both Pettit's and Chapman's arguments are concerned almost exclusively with the *procedural* merits of the premise- versus conclusion-based procedures. Bovens and Rabinowicz (2001a, 2001b), by contrast, have compared the two procedures from an *epistemic* perspective (see also Pettit and Rabinowicz 2001; on the distinction between procedural and epistemic conceptions of democracy, see List and Goodin 2001). Supposing – in the framework of the Condorcet jury theorem – that there is an independent fact of the matter on whether each of P and Q is true (and, by implication, on whether R is true), they study the likelihood that the premise- and conclusion-based procedures reach the correct decision on R. In this section, the Condorcet jury framework will be connected with the present probability-theoretic framework, and the implications of the Condorcet jury assumptions for the probability of collective inconsistencies will be discussed. I also present alternative proofs as well as extensions of some of the results by Bovens and Rabinowicz, particularly convergence results (some of them in appendix 3). In section 4 below, I provide generalizations of the results to a disjunctive version of the doctrinal paradox as well as to cases of more than two premises.

We assume (i) that each individual has probabilities p and q of making a correct judgment on P and on Q, respectively, where p, $q > 0.5$ (informally, these probabilities are interpreted as the "competence" of the individual); (ii) each individual's judgments on P and on Q are independent from each other; (iii) the judgments of different individuals are independent from each other. Again, these assumptions are in the spirit of the classical Condorcet jury theorem. At the end of this section we briefly address the effects of dependencies between the same individual's judgments on P and on Q.

Suppose the truth-values of P and Q are fixed (though not necessarily known). Then the values of p and q induce corresponding values of p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$. In other words, from the probabilities corresponding to each individual's decisions on P and Q, we can infer the probabilities corresponding to each individual's holding each of the sets of judgments PQ, $P\neg Q$, $\neg PQ$, $\neg P\neg Q$. The four possible cases are shown in table 4.

Table 4: p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$ as derived from p and q

	P	Q	p_{PQ}	$p_{P\neg Q}$	$p_{\neg PQ}$	$p_{\neg P\neg Q}$
Case 1	true	true	pq	$p(1-q)$	$(1-p)q$	$(1-p)(1-q)$
Case 2	true	false	$p(1-q)$	pq	$(1-p)(1-q)$	$(1-p)q$
Case 3	false	true	$(1-p)q$	$(1-p)(1-q)$	pq	$p(1-q)$
Case 4	false	false	$(1-p)(1-q)$	$(1-p)q$	$p(1-q)$	pq

Proposition 3. *Let the connection rule be* $(R \leftrightarrow (P \wedge Q))$.

(a) *Suppose P and Q are true.*

- *Suppose* $0.5 < p, q < \sqrt{(0.5)}$. *Then the probability of a collective inconsistency under propositionswise majority voting converges to* 1 *as n tends to infinity.*
- *Suppose* $p, q > \sqrt{(0.5)}$. *Then the probability of a collective inconsistency under propositionwise majority voting converges to* 0 *as n tends to infinity.*

(b) *Suppose that not both P and Q are true and* $p, q > 0.5$. *Then the probability of a collective inconsistency under propositionswise majority voting converges to* 0 *as n tends to infinity.*

See appendix 3 for a proof. Proposition 3 shows that convergence of the probability of the paradox to certainty (as the number of individuals increases) occurs when all premises are true and individual competence is better than random but not particularly high. Convergence of the probability of the paradox to 0 occurs when either at least one of the premises is false or individual competence is very high. As the premise- and conclusion-based procedures will produce divergent outcomes precisely in those cases in which a collective inconsistency occurs, proposition 3 immediately implies that, when all premises are true and individual competence is low (but better than random), the probability of a discrepancy between the two procedures will also converge to certainty as the number of individuals increases. To the extent that cases of true premises and low competence are plausible, discrepancies between the two procedures may thus be frequent. This observation motivates the question of which of the two procedures we should use from the perspective of making the correct decision.

Bovens and Rabinowicz distinguish between reaching the truth for the right reasons, and reaching it regardless of reasons. Reaching the truth for the right reasons requires deducing the correct decision on the conclusion from correct decisions on each of the premises, whereas reaching the truth regardless of reasons includes the possibility of reaching the correct decision on the conclusion accidentally, while making a wrong decision on at least one of the premises. Which of the two truth-tracking criteria we regard as the more compelling one depends on the account of democracy we hold. Deliberative democrats or lawyers in the common law tradition stress the importance of giving public reasons underlying collective decisions (Pettit 2001 and Chapman 2002), and would hence endorse the criterion of reaching the truth for the right reasons. Pure epistemic democrats or pure consequentialists, by contrast, focus primarily on reaching correct outcomes reliably, irrespective of the underlying reasoning *process*, and would therefore endorse the criterion of reaching the truth regardless of reasons.

Table 5 shows the conditions, in terms of the present framework, under which the premise- and conclusion-based procedures reach the correct decision on R (i) regardless of reasons and (ii) for the right reasons, for different truth-values of P and Q.

Table 5: Conditions under which the premise- and conclusion-based procedures reach the correct decision on *R* (given ($R \leftrightarrow (P \wedge Q)$)) (i) regardless of reasons and (ii) for the right reasons, for different truth-values of *P* and *Q*

<table>
<tr><td rowspan="2">P</td><td rowspan="2">Q</td><td colspan="2">Premise-based procedure reaches correct decision on R</td><td colspan="2">Conclusion-based procedure reaches correct decision on R</td></tr>
<tr><td>regardless of reasons
if and only if …</td><td>for the right reasons
if and only if …</td><td>regardless of reasons
if and only if …</td><td>for the right reasons
if and only if …</td></tr>
<tr><td>true</td><td>true</td><td colspan="2">there are majorities for each of P and Q
i.e.
$(n_{PQ} + n_{P\neg Q} > n/2)$ and $(n_{PQ} + n_{\neg PQ} > n/2)$
(1)</td><td colspan="2">there is a single majority supporting both P and Q
i.e.
$n_{PQ} > n/2$
(2)</td></tr>
<tr><td>true</td><td>false</td><td rowspan="3">there are not majorities for each of P and Q
i.e.
$(n_{PQ} + n_{P\neg Q} < n/2)$ or $(n_{PQ} + n_{\neg PQ} < n/2)$
(3)</td><td>there is a majority for P and a majority against Q
i.e.
$(n_{PQ} + n_{P\neg Q} > n/2)$ and $(n_{P\neg Q} + n_{\neg P\neg Q} > n/2)$
(4)</td><td rowspan="3">there is not a single majority supporting both P and Q
i.e.
$n_{PQ} < n/2$
(7)</td><td>there is a single majority supporting P and rejecting Q
i.e.
$n_{P\neg Q} > n/2$
(8)</td></tr>
<tr><td>false</td><td>true</td><td>there is a majority against P and a majority for Q
i.e.
$(n_{\neg PQ} + n_{\neg P\neg Q} > n/2)$ and $(n_{PQ} + n_{\neg PQ} > n/2)$
(5)</td><td>there is a single majority rejecting P and supporting Q
i.e.
$n_{\neg PQ} > n/2$
(9)</td></tr>
<tr><td>false</td><td>false</td><td>there are majorities against each of P and Q
i.e.
$(n_{\neg PQ}+n_{\neg P\neg Q}>n/2)$ and $(n_{P\neg Q} + n_{\neg P\neg Q} > n/2)$
(6)</td><td>there is a single majority rejecting both P and Q
i.e.
$n_{\neg P\neg Q} > n/2$
(10)</td></tr>
</table>

Bovens and Rabinowicz show in detail that the premise-based procedure is always better at reaching the correct decision on *R* for the right reasons, whereas the conclusion-based procedure may sometimes be better at reaching it regardless of reasons. Some of these results can be derived from table 5.

- Suppose we are concerned with reaching the correct decision on *R* for the right reasons. To compare the two procedures, we need to compare the relevant conditions corresponding to the four logically possible combinations of truth-values on *P* and *Q*. Condition (2) implies condition (1), condition (8) implies condition (4), condition (9) implies condition (5), and condition (10) implies condition (6). Hence the premise-based procedure is always at least as good as the conclusion-based procedure in terms of reaching the correct decision on R for the right reasons.

- Suppose we are concerned with reaching the correct decision on R regardless of reasons. Here we need to distinguish two cases.
    - Suppose both P and Q are true. Again, condition (2) implies condition (1), and hence the premise-based procedure is always at least as good as the conclusion-based procedure in terms of reaching the correct decision on R regardless of reasons.
    - Suppose not both P and Q are true. Here condition (3) implies condition (7), and hence the conclusion-based procedure is always at least as good as the premise-based procedure in terms of reaching the correct decision on R regardless of reasons.

These results are compatible with results by Grofman (1985) showing that, when a group decision on a conjunctive composite proposition can be disaggregated into separate group decisions on each of the conjuncts, disaggregation is superior in terms of reaching the correct decision (regardless of reasons) for true propositions, but not for false decisions.

Appendix 2 shows, in terms of the present framework, how to calculate the probabilities that, for a fixed number of individuals n and fixed truth-values of P and Q, the premise- and conclusion-based procedures reach the correct decision on R (i) regardless of reasons and (ii) for the right reasons.

The results by Bovens and Rabinowicz also imply several results on the convergence of these probabilities as the number of individuals increases. The present framework provides alternative proofs of some of these results, given in appendix 3.

Proposition 4. *Let the connection rule be* $(R \leftrightarrow (P \wedge Q))$. *The probabilities, as n tends to infinity, that the premise- and conclusion-based procedures reach a correct decision on R (i) regardless of reasons and (ii) for the right reasons, under various scenarios, are as shown in table* 6.

Table 6: Probability, as *n* tends to infinity, of a correct decision on *R* (given ($R \leftrightarrow (P \wedge Q)$)) under the premise- and conclusion based procedures (i) regardless of reasons and (ii) for the right reasons, under various scenarios

<table>
<tr><th rowspan="2"></th><th colspan="2">Premise-based procedure:
Probability, as n tends to infinity, of …</th><th colspan="2">Conclusion-based procedure:
Probability, as n tends to infinity, of …</th></tr>
<tr><th>a correct decision on R regardless of reasons</th><th>a correct decision on R for the right reasons</th><th>a correct decision on R regardless of reasons</th><th>a correct decision on R for the right reasons</th></tr>
<tr><td>$0.5 < p, q < \sqrt{(0.5)}$
P and Q both true</td><td colspan="2" rowspan="3">1
(a)</td><td colspan="2">0
(b)</td></tr>
<tr><td>$0.5 < p, q < \sqrt{(0.5)}$
not both
P and Q true</td><td>1
(c)</td><td>0
(d)</td></tr>
<tr><td>$p, q > \sqrt{(0.5)}$</td><td colspan="2">1
(e)</td></tr>
</table>

For a large class of conditions, the performance of the conclusion-based procedure is poor. If we are concerned with tracking the "truth" for the right reasons, the probability that the conclusion-based procedure will be successful will always converge to 0 as the number of individuals increases, unless the competence of individuals exceeds $\sqrt{(0.5)}$. If we are concerned with tracking the "truth" regardless of reasons, then the probability that the conclusion-based procedure will be successful will still converge to 0, unless at least one of the premises is false. By contrast, the probability that the premise-based procedure tracks the "truth", both for the right reasons and regardless of reasons, will converge to 1 as soon as the competence of individuals is above 0.5. However, when *P* and *Q* are not both true, then the probability that the conclusion-based procedure reaches the correct decision on *R* regardless of reasons converges to 1 *faster* than the probability that the premise-based procedure reaches the correct decision on *R* regardless of reasons. This follows from the fact (remarked above) that condition (3) in table 5 implies condition (7), whereas the converse implication does not hold.

It is important to note that the results of this section are very much dependent on the assumption that each individual's judgments on *P* and on *Q* are independent from each other. If there is a high degree of dependency, the probability of a collective inconsistency and of a discrepancy between the premise- and conclusion-based procedures is drastically reduced. In the limiting case, if each individual makes a correct judgment on *P if and only if* he or she makes a correct judgment on *Q*, then the individual's competence on the conclusion *R* is equal to his or her competence on each of the premises P and Q. And, so long as that competence exceeds 0.5, the classical Condorcet jury theorem can then be applied to show that the probability of a correct decision under the conclusion-based procedure converges to 1 as the number of individuals increases. In this case of *perfect* dependency between an individual's judgments on P and on Q, the premise- and conclusion-based procedures will always coincide and there will be no collective inconsistencies.

4 Extensions and Generalizations

So far we have discussed only one specific version of a problem of aggregation over multiple interconnected propositions, namely the conjunctive version of the doctrinal paradox, where the conjunction of two premises is a necessary and sufficient condition for a conclusion. It is known that the paradox can be generalized. Disjunctive versions of the paradox have been discussed, as well as extensions to more than two propositions (see, amongst others, Chapman 1998 and Pettit 2001). Moreover, for *any* system of multiple propositions with certain logical interconnections, collective inconsistencies under propositionwise majority voting are possible (List and Pettit 2002a). The aim of the present section is to illustrate that the present method of determining the probability of collective inconsistencies under propositionwise majority voting is applicable to other problems of aggregation over multiple propositions too. I will discuss two applications of the method, first an application to the disjunctive version of the doctrinal paradox, and second an application to the conjunctive version of the paradox with more than two premises.

4.1 The Disjunctive Version of the Doctrinal Paradox

Table 7: The Doctrinal Paradox (Disjunctive Version)

	P	Q	$(R \leftrightarrow (P \vee Q))$	R
Judge 1	Yes	No	Yes	Yes
Judge 2	No	Yes	Yes	Yes
Judge 3	No	No	Yes	No
Majority	No	No	Yes	Yes

In the disjunctive version of the doctrinal paradox, there are two premises, P and Q (e.g. "there is possibility 1 for jurisdiction" and "there is possibility 2 for jurisdiction"), and a conclusion, R ("there is a possibility for jurisdiction, all things considered"), and all judges accept that the disjunction of P and Q is necessary and sufficient for R. Given the individual judgments in table 7, a majority *accepts* R, a majority (unanimity) *accepts* ($R \leftrightarrow (P \vee Q)$), and yet majorities *reject* each of P and Q. A majority of judges may, for example, hold that there is a possibility for jurisdiction, all things considered, and also that this possibility for jurisdiction must be supported by at least one of two justifications (P or Q), but the judges may feel to reach any majority agreement on which of the two justifications obtains.

Once again, there are 4 logically possible consistent sets of judgments an individual might hold, as shown in table 8.

Table 8: All logically possible consistent sets of judgments over *P*, *Q* and *R*, given ($R \leftrightarrow (P \vee Q)$)

Label	Judgment on P	Judgment on Q	Judgment on R
PQ	Yes	Yes	Yes
$P\neg Q$	Yes	No	Yes
$\neg PQ$	No	Yes	Yes
$\neg P\neg Q$	No	No	No

Note that the connection rule ($R \leftrightarrow (P \vee Q)$) is logically equivalent to ($\neg R \leftrightarrow (\neg P \wedge \neg Q)$). Therefore all the results on the conjunctive version of the paradox in section 2 can be restated for the disjunctive version too. To state the corresponding results for the disjunctive version of the paradox, we simply need to swap P and $\neg P$, Q and $\neg Q$ and R and $\neg R$ in all the propositions and proofs.

The following proposition is the counterpart of propositions 1 and 2 above. Let n_{PQ}, $n_{P\neg Q}$, $n_{\neg PQ}$, $n_{\neg P\neg Q}$ be the numbers of individuals holding the sets of judgments PQ, $P\neg Q$, $\neg PQ$, $\neg P\neg Q$ in table 8, respectively.

Proposition 5. *Let the connection rule be* ($R \leftrightarrow (P \vee Q)$).

(a) *There will be a collective inconsistency under propositionwise majority voting if and only if* ($n_{\neg P\neg Q} + n_{\neg PQ} > n/2$) *and* ($n_{\neg P\neg Q} + n_{P\neg Q} > n/2$) *and* ($n_{\neg P\neg Q} < n/2$).

(b) *Suppose* ($p_{\neg P\neg Q} + p_{\neg PQ} > 1/2$) *and* ($p_{\neg P\neg Q} + p_{P\neg Q} > 1/2$) *and* ($p_{\neg P\neg Q} < 1/2$). *Then the probability of a collective inconsistency under propositionwise majority voting converges to* 1 *as n tends to infinity.*

(c) *Suppose* ($p_{\neg P\neg Q} + p_{\neg PQ} < 1/2$) *or* ($p_{\neg P\neg Q} + p_{P\neg Q} < 1/2$) *or* ($p_{\neg P\neg Q} > 1/2$). *Then the probability of a collective inconsistency under propositionwise majority voting converges to* 0 *as n tends to infinity.*

Given unanimous acceptance of $(R \leftrightarrow (P \vee Q))$, the conditions of 5a are necessary and sufficient for the majority acceptance of the (inconsistent) set of propositions $\{\neg P, \neg Q, (R \leftrightarrow (P \vee Q)), R\}$. The proofs of 5b and 5c are are analogous to the proofs of 2a and 2b.

If scenarios 1 to 8 in table 3 are replaced with scenarios 1* to 8*, as shown in table 9 below, the probability that there will be a collective inconsistency under propositionwise majority voting for the new connection rule $(R \leftrightarrow (P \vee Q))$ can be read off directly from table 3.

Table 9: Scenarios corresponding to the probability that there will be a collective inconsistency under propositionwise majority voting (given $(R \leftrightarrow (P \vee Q))$)

Scenario 1*	Scenario 2*	Scenario 3*	Scenario 4*	Scenario 5*	Scenario 6*	Scenario 7*	Scenario 8*
$p_{PQ} = 0.25$	$p_{PQ} = 0.24$	$p_{PQ} = 0.2$	$p_{PQ} = 0.24$	$p_{PQ} = 0.11$	$p_{PQ} = 0.09$	$p_{PQ} = 0.05$	$p_{PQ} = 0.01$
$p_{P\neg Q} = 0.25$	$p_{P\neg Q} = 0.25$	$p_{P\neg Q} = 0.25$	$p_{P\neg Q} = 0.25$	$p_{P\neg Q} = 0.2$	$p_{P\neg Q} = 0.2$	$p_{P\neg Q} = 0.2$	$p_{P\neg Q} = 0.33$
$p_{\neg PQ} = 0.25$	$p_{\neg PQ} = 0.25$	$p_{\neg PQ} = 0.25$	$p_{\neg PQ} = 0.27$	$p_{\neg PQ} = 0.2$	$p_{\neg PQ} = 0.2$	$p_{\neg PQ} = 0.2$	$p_{\neg PQ} = 0.33$
$p_{\neg P\neg Q} = 0.25$	$p_{\neg P\neg Q} = 0.26$	$p_{\neg P\neg Q} = 0.3$	$p_{\neg P\neg Q} = 0.24$	$p_{\neg P\neg Q} = 0.49$	$p_{\neg P\neg Q} = 0.51$	$p_{\neg P\neg Q} = 0.55$	$p_{\neg P\neg Q} = 0.33$

The conditions of proposition 5b – convergence of the probability of a collective inconsistency to 1 – are satisfied in scenarios 2*, 3*, 5* and 8*; the conditions of proposition 5c – convergence of the probability of a collective inconsistency to 0 – are satisfied in scenarios 4*, 6* and 7*.

We will now again use the Condorcet jury framework introduced in section 3.

Proposition 6. *Let the connection rule be* $(R \leftrightarrow (P \vee Q))$.

(a) *Suppose P and Q are both false.*

- *Suppose* $0.5 < p, q < \sqrt{(0.5)}$. *Then the probability of a collective inconsistency under propositionswise majority voting converges to* 1 *as n tends to infinity.*
- *Suppose* $p, q > \sqrt{(0.5)}$. *Then the probability of a collective inconsistency under propositionwise majority voting converges to* 0 *as n tends to infinity.*

(b) *Suppose that at least one of P and Q is true and* $p, q > 0.5$. *Then the probability of a collective inconsistency under propositionswise majority voting converges to* 0 *as n tends to infinity.*

The premise- and conclusion-based procedures of decision-making provide escape-routes from the disjunctive version of the doctrinal paradox too. In analogy to proposition 3, proposition 6 can be interpreted as showing that the probability of a discrepancy between the two procedures (which occurs precisely when the paradox occurs) will converge to certainty (as the number of individuals increases) when both premises are false and individual competence is low (but better than random). Again, this motivates the question of which of the two procedures performs better from the perspective of truth-tracking. As in the conjunctive case, we can distinguish between reaching the correct decision for the right reasons and reaching it regardless of reasons. Table 10 shows the conditions under which the premise- and conclusion-based procedures reach the correct decision on R (given $(R \leftrightarrow (P \vee Q))$) (i) regardless of reasons and (ii) for the right reasons, for different truth-values of P and Q.

Table 10: Conditions under which the premise- and conclusion-based procedures reach the correct decision on *R* (given $(R \leftrightarrow (P \vee Q))$) (i) regardless of reasons and (ii) for the right reasons, for different truth-values of *P* and *Q*

<table>
<tr><td rowspan="2">P</td><td rowspan="2">Q</td><td colspan="2">Premise-based procedure reaches correct decision on R</td><td colspan="2">Conclusion-based procedure reaches correct decision on R</td></tr>
<tr><td>regardless of reasons
if and only if …</td><td>for the right reasons
if and only if …</td><td>regardless of reasons
if and only if …</td><td>for the right reasons
if and only if …</td></tr>
<tr><td>false</td><td>false</td><td colspan="2">there are majorities against each of P and Q
i.e.
$(n_{\neg P\neg Q} + n_{P\neg Q} > n/2)$ and $(n_{\neg P\neg Q} + n_{\neg PQ} > n/2)$
(1)</td><td colspan="2">there is a single majority against both P and Q
i.e.
$n_{\neg P\neg Q} > n/2$
(2)</td></tr>
<tr><td>true</td><td>false</td><td rowspan="3">there is a majority for at least one of P and Q
i.e.
$(n_{PQ} + n_{P\neg Q} > n/2)$ or $(n_{PQ} + n_{\neg PQ} > n/2)$
(3)</td><td>there is a majority for P and a majority against Q
i.e.
$(n_{PQ} + n_{P\neg Q} > n/2)$ and $(n_{P\neg Q} + n_{\neg P\neg Q} > n/2)$
(4)</td><td rowspan="3">there is not a single majority against both P and Q
i.e.
$n_{\neg P\neg Q} < n/2$
(7)</td><td>there is a single majority supporting P and rejecting Q
i.e.
$n_{P\neg Q} > n/2$
(8)</td></tr>
<tr><td>false</td><td>true</td><td>there is a majority against P and a majority for Q
i.e.
$(n_{\neg PQ} + n_{\neg P\neg Q} > n/2)$ and $(n_{PQ} + n_{\neg PQ} > n/2)$
(5)</td><td>there is a single majority rejecting P and supporting Q
i.e.
$n_{\neg PQ} > n/2$
(9)</td></tr>
<tr><td>true</td><td>true</td><td>there are majorities for each of P and Q
i.e.
$(n_{\neg PQ}+n_{PQ}>n/2)$ and $(n_{P\neg Q} + n_{PQ} > n/2)$
(6)</td><td>there is a single majority supporting both P and Q
i.e.
$n_{PQ} > n/2$
(10)</td></tr>
</table>

In table 10, the same implications as in table 5 hold, and we can deduce the following propositions:

- The premise-based procedure is always at least as good as the conclusion-based procedure in terms of reaching the correct decision on *R* for the right reasons.
- Suppose we are concerned with reaching the correct decision on *R* regardless of reasons. Here we need to distinguish two cases.
    - Suppose both *P* and *Q* are false. Then the premise-based procedure is always at least as good as the conclusion-based procedure in terms of reaching the correct decision on *R* regardless of reasons.
    - Suppose at least one of *P* and *Q* is true. Then the conclusion-based procedure is always at least as good as the premise-based procedure in terms of reaching the correct decision on *R* regardless of reasons.

These results are also compatible with results by Grofman (1985). Grofman showed that, when a group decision on a disjunctive composite proposition can be disaggregated into separate group decisions on each of the disjuncts, disaggregation is superior in terms of reaching the correct decision (regardless of reasons) for false propositions, but not for true decisions.

Proposition 7. *Let the connection rule be* ($R \leftrightarrow (P \vee Q)$). *The probabilities, as n tends to infinity, that the premise- and conclusion-based procedures reach a correct decision on R (i) regardless of reasons and (ii) for the right reasons, under various scenarios, are as shown in table* 11.

Table 11: Probability, as *n* tends to infinity, of a correct decision on *R* (given ($R \leftrightarrow (P \vee Q)$)) under the premise- and conclusion based procedures (i) regardless of reasons and (ii) for the right reasons, under various scenarios

<table>
<tr><th rowspan="2"></th><th colspan="2">Premise-based procedure:
Probability, as n tends to infinity, of …</th><th colspan="2">Conclusion-based procedure:
Probability, as n tends to infinity, of …</th></tr>
<tr><th>a correct decision on R regardless of reasons</th><th>a correct decision on R for the right reasons</th><th>a correct decision on R regardless of reasons</th><th>a correct decision on R for the right reasons</th></tr>
<tr><td>$0.5 < p, q < \sqrt{(0.5)}$
P and Q both false</td><td colspan="2" rowspan="3">1
(a)</td><td colspan="2">0
(b)</td></tr>
<tr><td>$0.5 < p, q < \sqrt{(0.5)}$
at least one of
P and Q true</td><td>1
(c)</td><td>0
(d)</td></tr>
<tr><td>$p, q > \sqrt{(0.5)}$</td><td colspan="2">1
(e)</td></tr>
</table>

As in the conjunctive case, for a large class of conditions, the performance of the conclusion-based procedure is poor, particularly if we are concerned with tracking the "truth" for the right reasons. Unlike in the conjunctive case, however, the probability that the conclusion-based procedure will track the "truth" regardless of reasons converges to 1 if at least one of the premises is true. Here, when P and Q are not both false, then the probability that the conclusion-based procedure reaches the correct decision on R regardless of reasons converges to 1 *faster* than the probability that the premise-based procedure reaches the correct decision on R regardless of reasons. This follows from the fact that condition (3) in table 10 implies condition (7), whereas the converse implication does not hold (compare the remarks on the conjunctive case in section 3 above).

4.2 The Conjunctive Version of the Doctrinal Paradox with More than Two Premises

First we will generalize propositions 1 and 2 to the case of three premises. We will then generalize propositions 3 and 4 to the case of k premises.

Table 12: The Doctrinal Paradox (The Case of Three-Premises)

	P	Q	R	$(S \leftrightarrow (P \wedge Q \wedge R))$	S
Individual 1	Yes	Yes	No	Yes	No
Individual 2	No	Yes	Yes	Yes	No
Individual 3	Yes	No	Yes	Yes	No
Majority	Yes	Yes	Yes	Yes	No

If the individual judgments are as in table 12, there are propositionwise majorities for each of the three premises, P, Q and R; all individuals accept that the conjunction of the three premises is necessary and sufficient for the conclusion, S; and yet S is unanimously rejected.

This time there are 8 logically possible consistent sets of judgments an individual might hold, as shown in table 13.

Table 13: All logically possible consistent sets of judgments over *P*, *Q, R* and S, given (*S* ↔ (*P* ∧ *Q* ∧ *R*))

	P	Q	R	S
PQR	Yes	Yes	Yes	Yes
$PQ\neg R$	Yes	Yes	No	No
$P\neg QR$	Yes	No	Yes	No
$P\neg Q\neg R$	Yes	No	No	No
$\neg PQR$	No	Yes	Yes	No
$\neg PQ\neg R$	No	Yes	No	No
$\neg P\neg QR$	No	No	Yes	No
$\neg P\neg Q\neg R$	No	No	No	No

Let n_{PQR}, $n_{PQ\neg R}$, $n_{P\neg QR}$, $n_{P\neg Q\neg R}$, $n_{\neg PQR}$, $n_{\neg PQ\neg R}$, $n_{\neg P\neg QR}$, $n_{\neg P\neg Q\neg R}$ be the numbers of individuals holding the sets of judgments in table 13, and let p_{PQR}, $p_{PQ\neg R}$, $p_{P\neg QR}$, $p_{P\neg Q\neg R}$, $p_{\neg PQR}$, $p_{\neg PQ\neg R}$, $p_{\neg P\neg QR}$, $p_{\neg P\neg Q\neg R}$ be the corresponding probabilities.

Proposition 8. *Let the connection rule be* $(S \leftrightarrow (P \wedge Q \wedge R))$.

(a) *There will be a collective inconsistency under propositionwise majority voting if and only if* $(n_{PQR}+n_{PQ\neg R}+n_{P\neg QR},+n_{P\neg Q\neg R} > n/2)$ *and* $(n_{PQR}+n_{PQ\neg R}+n_{\neg PQR}+n_{\neg PQ\neg R} > n/2)$ *and* $(n_{PQR}+n_{P\neg QR}+n_{\neg PQR}+n_{\neg P\neg QR} > n/2)$ *and* $(n_{PQR} < n/2)$.

(b) *Suppose* $(p_{PQR}+p_{PQ\neg R}+p_{P\neg QR},+p_{P\neg Q\neg R} > 1/2)$ *and* $(p_{PQR}+p_{PQ\neg R}+p_{\neg PQR}+p_{\neg PQ\neg R} > 1/2)$ *and* $(p_{PQR}+p_{P\neg QR}+p_{\neg PQR}+p_{\neg P\neg QR} > 1/2)$ *and* $(p_{PQR} < 1/2)$. *Then the probability of a collective inconsistency under propositionwise majority voting converges to* 1 *as n tends to infinity.*

(c) *Suppose* $(p_{PQR}+p_{PQ\neg R}+p_{P\neg QR},+p_{P\neg Q\neg R} < 1/2)$ *or* $(p_{PQR}+p_{PQ\neg R}+p_{\neg PQR}+p_{\neg PQ\neg R} < 1/2)$ *or* $(p_{PQR}+p_{P\neg QR}+p_{\neg PQR}+p_{\neg P\neg QR} < 1/2)$ *or* $(p_{PQR} > 1/2)$. *Then the probability of a collective inconsistency under propositionwise majority voting converges to* 0 *as n tends to infinity.*

The proof of proposition 8 is given in appendix 3. To illustrate, the conditions of proposition 8b – convergence of the probability of a collective inconsistency to 1 – are satisfied when $p_{PQR} = 0.126$, $p_{\neg P\neg Q\neg R} = 0.124$ and $p_{PQ\neg R} = p_{P\neg QR} = p_{P\neg Q\neg R} = p_{\neg PQR} = p_{\neg PQ\neg R} = p_{\neg P\neg QR} = 0.125$; or when $p_{PQR} = 0.49$, $p_{\neg P\neg Q\neg R} = 0.03$ and $p_{PQ\neg R} = p_{P\neg QR} = p_{P\neg Q\neg R} = p_{\neg PQR} = p_{\neg PQ\neg R} = p_{\neg P\neg QR} = 0.08$. The conditions of proposition 8c – convergence of the probability of a collective inconsistency to 0 – are satisfied when $p_{PQR} = 0.124$, $p_{\neg P\neg Q\neg R} = 0.126$ and $p_{PQ\neg R} = p_{P\neg QR} = p_{P\neg Q\neg R} = p_{\neg PQR} = p_{\neg PQ\neg R} = p_{\neg P\neg QR} = 0.125$; or when $p_{PQR} = 0.51$, $p_{\neg P\neg Q\neg R} = 0.01$ and $p_{PQ\neg R} = p_{P\neg QR} = p_{P\neg Q\neg R} = p_{\neg PQR} = p_{\neg PQ\neg R} = p_{\neg P\neg QR} = 0.08$.

We will now generalize propositions 3 and 4 to the case of k premises. This generalization will serve to illustrate how easily a collective inconsistency can occur when the number of propositions is large. We will consider an aggregation problem with k premises, $P_1, P_2, ..., P_k$, whose conjunction is necessary and sufficient for a conclusion, R. Again, we assume (i) that each individual has probabilities (individual "competence") $p_1, p_2, ..., p_k$ of making a correct judgment on $P_1, P_2, ..., P_k$, respectively, where $p_1, p_2, ..., p_k > 0.5$; (ii) each individual's judgments on $P_1, P_2, ..., P_k$ are independent from each other; (iii) the judgments of different individuals are independent from each other. The proofs of propositions 9 and 10 are perfectly analogous to the proofs of their counterparts for two premises (propositions 3 and 4 above). Note that the probability that an individual holds the conjunction of correct judgments on $P_1, P_2, ..., P_k$ is the product $p_1 p_2 ... p_k$. In particular, if $p_1, p_2, ..., p_k < {}^k\sqrt{(0.5)}$, then $p_1 p_2 ... p_k < 0.5$; and if $p_1, p_2, ..., p_k > {}^k\sqrt{(0.5)}$, then $p_1 p_2 ... p_k > 0.5$.

Proposition 9. *Let the connection rule be* $(R \leftrightarrow (P_1 \wedge P_2 \wedge ... \wedge P_k))$.

(a) *Suppose* $P_1, P_2, ..., P_k$ *are true.*

- *Suppose* $0.5 < p_1, p_2, ..., p_k < {}^k\sqrt{(0.5)}$. *Then the probability of a collective inconsistency under propositionswise majority voting converges to* 1 *as* n *tends to infinity.*
- *Suppose* $p, q > {}^k\sqrt{(0.5)}$. *Then the probability of a collective inconsistency under propositionwise majority voting converges to* 0 *as* n *tends to infinity.*

(b) *Suppose that not all of* $P_1, P_2, ..., P_k$ *are true and* $p_1, p_2, ..., p_k > 0.5$. *Then the probability of a collective inconsistency under propositionswise majority voting converges to* 0 *as* n *tends to infinity.*

Proposition 10. *The probabilities, as* n *tends to infinity, that the premise- and conclusion-based procedures reach a correct decision on R (i) regardless of reasons and (ii) for the right reasons, under various scenarios, are as shown in table* 14.

Table 14: Probability, as n tends to infinity, of a correct decision on R (given ($R \leftrightarrow (P_1 \wedge P_2 \wedge \ldots \wedge P_k)$)) under the premise- and conclusion based procedures (i) regardless of reasons and (ii) for the right reasons, under various scenarios

<table>
<tr><td rowspan="2"></td><td colspan="2">Premise-based procedure:
Probability, as n tends to infinity, of …</td><td colspan="2">Conclusion-based procedure:
Probability, as n tends to infinity, of …</td></tr>
<tr><td>a correct decision on R regardless of reasons</td><td>a correct decision on R for the right reasons</td><td>a correct decision on R regardless of reasons</td><td>a correct decision on R for the right reasons</td></tr>
<tr><td>$0.5 < p_1, p_2, \ldots, p_k < \sqrt[k]{0.5}$
$P_1, P_2, \ldots, P_k$ all true</td><td colspan="2" rowspan="3">1
(a)</td><td colspan="2">0
(b)</td></tr>
<tr><td>$0.5 < p_1, p_2, \ldots, p_k < \sqrt[k]{0.5}$
not all of $P_1, P_2, \ldots, P_k$ true</td><td>1
(c)</td><td>0
(d)</td></tr>
<tr><td>$p_1, p_2, \ldots, p_k > \sqrt[k]{0.5}$</td><td colspan="2">1
(e)</td></tr>
</table>

Several points can be noted from propositions 9 and 10. For a large number k of premises, the level of individual competence required for the avoidance of collective inconsistencies (when all premises are true) is very high; the requisite lower bound on each of $p_1, p_2, \ldots, p_k$, namely $\sqrt[k]{0.5}$, converges to 1 as k increases. Moreover, unless the competence of individuals is above that bound, the performance of the conclusion-based procedure in terms of reaching a correct decision on the conclusion for the right reasons is very poor. Moreover, if the premises are all true, the conclusion-based procedure will also perform poorly in terms of reaching a correct decision on the conclusion regardless of reasons. The premise-based procedure, by contrast, will reach a correct decision on the conclusion more reliably, both for the right reasons and regardless of reasons (a remark on the differential speed of convergence similar to the one in section 3 applies). Again, the results are dependent on the assumption that each individual's judgments on the different premises are independent from each other.

5 The Probability of Inconsistent Collective Sets of Judgments Compared with the Probability of Cycles

The doctrinal paradox invites comparison with Condorcet's paradox concerning voting over multiple options, according to which consistent individual preferences can lead to inconsistent collective preferences under pairwise majority voting (the parallels between the two paradoxes are discussed in List and Pettit 2002b). To state Condorcet's paradox, suppose there are three individuals, where one prefers option x_1 to option x_2 to option x_3, the second prefers option x_2 to option x_3 to option x_1, and the third prefers option x_3 to option x_1 to option x_2. Then there is a majority for x_1 against x_2, a majority for x_2 against x_3, and a majority for x_3 against x_1, a cycle.

Several recent papers have addressed the likelihood of the occurrence of Condorcet's paradox in a large electorate (Tangian 2000; Tsetlin, Regenwetter and Grofman 2000; List and Goodin 2001). The robust finding is that, given plausible assumptions about the distribution of individual preferences, the probability of cyclical collective preferences vanishes as the number of individuals increases. In what follows, I will briefly discuss the parallels between existing results on the probability of cycles and the present results on the probability of inconsistent collective sets of judgements, using the example of the conjunctive version of the doctrinal paradox.

We have seen in section 2 that slight deviations from an impartial culture can imply convergence of the probability of collective inconsistencies under propositionwise majority voting to either 0 or 1 as the number of individuals increases, depending on the precise pattern of deviation. A similar result holds for the aggregation of preferences.

If there are three options, x_1, x_2 and x_3, there are 6 logically possible strict preference orderings, as shown in table 15.

Table 15: All logically possible strict preference orderings over three options

Label	1st preference	2nd preference	3rd preference
P_{X1}	x_3	x_1	x_2
P_{Y2}	x_3	x_2	x_1
P_{Z1}	x_2	x_3	x_1
P_{X2}	x_2	x_1	x_3
P_{Y1}	x_1	x_2	x_3
P_{Z2}	x_1	x_3	x_2

Let p_{X1}, p_{X2}, p_{Y1}, p_{Y2}, p_{Z1}, p_{Z2} be the probabilities that an individual holds the orderings P_{X1}, P_{X2}, P_{Y1}, P_{Y2}, P_{Z1}, P_{Z2}, respectively (where the sum of the probabilities is 1). As before, an *impartial culture* is the situation in which $p_{X1} = p_{X2} = p_{Y1} = p_{Y2} = p_{Z1} = p_{Z2}$.

In an impartial culture, the probability of a cycle increases as the number of individuals increases (Gehrlein 1983). But, as in the case of the doctrinal paradox, an impartial culture is a special case (see in particular Tsetlin, Regenwetter and Grofman 2000). Given suitable systematic, however slight, deviations from an impartial culture, the probability of a cycle under pairwise majority voting will converge to either 0 or 1 as the number of individuals increases.

Proposition 11 (List 2001). *Let* $\delta = |p_{X1} - p_{X2}| + |p_{Y1} - p_{Y2}| + |p_{Z1} - p_{Z2}|$.

(a) *Suppose* $((p_{X1} > p_{X2}$ *and* $p_{Y1} > p_{Y2}$ *and* $p_{Z1} > p_{Z2})$ *or* $(p_{X1} < p_{X2}$ *and* $p_{Y1} < p_{Y2}$ *and* $p_{Z1} < p_{Z2}))$ *and* $(|p_{X1} - p_{X2}| < \delta/2)$ *and* $(|p_{Y1} - p_{Y2}| < \delta/2)$ *and* $(|p_{Z1} - p_{Z2}| < \delta/2)$. *Then the probability of a cycle under pairwise majority voting converges to 1 as n tends to infinity.*

(b) *Suppose* $((p_{X1} < p_{X2}$ *or* $p_{Y1} < p_{Y2}$ *or* $p_{Z1} < p_{Z2})$ *and* $(p_{X1} > p_{X2}$ *or* $p_{Y1} > p_{Y2}$ *or* $p_{Z1} > p_{Z2}))$ *or* $(|p_{X1} - p_{X2}| > \delta/2)$ *or* $(|p_{Y1} - p_{Y2}| > \delta/2)$ *or* $(|p_{Z1} - p_{Z2}| > \delta/2)$. *Then the probability of a cycle under pairwise majority voting converges to* 0 *as n tends to infinity.*

Propositions 11a and 11b correspond, respectively, to propositions 2a and 2b above. Proposition 11a, like proposition 2a, states conditions under which the probability of an inconsistent (here cyclical) outcome converges to 1. Proposition 11b, like proposition 2b, states conditions under which that probability converges to 0.

Thus, in the cases of both the probability of cycles and the probability of inconsistent collective sets of judgments, an impartial culture is a special case, implying a non-zero probability of the paradox. Further, in both cases, systematic deviations from an impartial culture imply convergence of that probability to either 0 or 1. Can we nonetheless find a criterion for determining whether the occurrence of one of the two paradoxes is empirically more likely than that of the other? The criterion would have to determine what distributions of probabilities over all logically possible preference orderings, or over all logically possible sets of judgments, are empirically plausible. We would then have to ask, in the case of the doctrinal paradox, whether these distributions satisfy the conditions of proposition 2a or those of proposition 2b, and in the case of Condorcet's paradox, whether they satisfy the conditions of proposition 11a or those of proposition 11b.

An initial inspection suggests that both the conditions of proposition 2a and those of proposition 2b can easily be met. For instance, the conditions of proposition 2a – convergence of the probability of a collective inconsistency to 1 – are already met if $p_{PQ} = 1/4 + \varepsilon$, $p_{\neg P \neg Q} = 1/4 - \varepsilon$ and $p_{\neg PQ} = p_{P\neg Q} = 1/4$, for any arbitrarily small number $\varepsilon > 0$. The conditions of proposition 2b – convergence of that probability to 0 – are already met if $p_{PQ} = 1/4 - \varepsilon$, $p_{\neg P \neg Q} = 1/4 + \varepsilon$ and $p_{\neg PQ} = p_{P\neg Q} = 1/4$. By contrast, the conditions of proposition 11b – convergence of the probability of cycles to 0 – appear to be logically less demanding than those of proposition 11a – convergence of that probability to 1. While the former are already met if *at least one* of $p_{X1} < p_{X2}$, $p_{Y1} < p_{Y2}$, $p_{Z1} < p_{Z2}$ and *at least one* of $p_{X1} > p_{X2}$, $p_{Y1} > p_{Y2}$, $p_{Z1} > p_{Z2}$ hold, the latter would require *all* of ($p_{X1} < p_{X2}$ *and* $p_{Y1} < p_{Y2}$ *and* $p_{Z1} < p_{Z2}$) or *all* of ($p_{X1} > p_{X2}$ *and* $p_{Y1} > p_{Y2}$ *and* $p_{Z1} > p_{Z2}$) *and* three additional conjuncts. For instance, the conditions of proposition 11b are already satisfied if $p_{X1} = 1/6 - \varepsilon$, $p_{Y1} = 1/6 + \varepsilon$ and $p_{X2} = p_{Y2} = p_{Z1} = p_{Z2} = 1/6$, while no equally simple deviation from an impartial culture is sufficient for the conditions of proposition 11a.

A more sophisticated *a priori* method of comparing the probabilities of the two paradoxes would be to compare [the volume (in $\mathbf{R}^6$) of the set of all probability vectors satisfying condition 11a divided by the volume (in $\mathbf{R}^6$) of the set of all possible probability vectors] and [the volume (in $\mathbf{R}^4$) of the set of all probability vectors satisfying condition 2a divided by the volume (in $\mathbf{R}^4$) of the set of all possible probability vectors]. However, the interpretation of such a comparison is not straightforward, and an *a priori* inspection of the conditions alone can hardly settle the question of whether the occurrence of one of the two paradoxes is *empirically* more likely than that of the other.

Nonetheless, to identify one criterion by which we might break the apparent correspondence between conditions 11a and 2a and between conditions 11b and 2b, we will again invoke a Condorcet jury framework. In the cases of both voting over multiple options and aggregation over multiple interconnected propositions, we will consider a suitable minimal Condorcet-jury-like competence assumption and then ask whether this assumption implies convergence of the probability of the relevant paradox – respectively, Condorcet's paradox and the doctrinal paradox – to 0 or to 1.

First, let us state the minimal competence assumption in the case of voting over multiple options. We suppose there are k options, $x_1, x_2, ..., x_k$, where (i) each individual has probabilities $p_1, p_2, ..., p_k$ of choosing $x_1, x_2, ..., x_k$ as their first choice, respectively, and (ii) the preferences of different individuals are independent from each other. We consider the special case $k = 3$.

Minimal Competence Assumption C1: Voting over Multiple Options. If x_j is the "correct" option, then, for all i (where $i \neq j$), $p_j > p_i$.

Further, supposing that the correct option is fixed, the values of p_1, p_2 and p_3 given by assumption C1 can be used to induce values of p_{X1}, p_{X2}, p_{Y1}, p_{Y2}, p_{Z1}, p_{Z2}, corresponding to each individual's holding each of the 6 logically possible strict preference orderings.

Specifically, we define the probability for the strict ordering $x_{i_1} > x_{i_2} > x_{i_3}$ (where i_1, i_2, i_3 $\in$ {1, 2, 3}) to be $p_{i_1} p_{i_2} / (1-p_{i_1})$ (see List and Goodin 2001). This corresponds to the way in which the values of p and q induce values of p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$, as discussed in section 3.

Assumption C1 implies that the probability distribution over all logically possible strict preference orderings is skewed, however slightly, in favour of preference orderings which rank the “correct” option above the other options. Note, however, that in such a skewed distribution there may still be a level of preference diversity that is arbitrarily close to an impartial culture. It is thus a "minimal" competence assumption, in so far as it can be satisfied in any ε-neighbourhood of an impartial culture.

In the case of aggregation over multiple interconnected propositions, the minimal competence assumption is given by the Condorcet jury framework introduced in section 3.

Minimal Competence Assumption C2: Aggregation over Multiple Interconnected Propositions. Each individual has probabilities p and q of making a correct judgment on P and Q, respectively, where $0.5 < p, q < \sqrt{(0.5)}$.

As we have seen in section 3, the values of p and q induce values of p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$.

Assumption C2 implies that the probability distribution over all logically possible individual sets of judgments is skewed, however slightly, in favour of the “correct” judgment on each premise. Assumption C2 is also a "minimal" competence assumption, in that it can be satisfied in any ε-neighbourhood of an impartial culture. Assumption C2 imposes not only a lower bound, 0.5, but also an upper bound, $\sqrt{(0.5)}$, on the individuals’ probabilities of making correct judgments on the premises. A similar (or even more stringent) upper bound may be imposed on individual competence in assumption C1 without affecting any of the results stated here.

We are now in a position to compare the implications of assumption C1 with those of assumption C2. In the case of voting over multiple options, C1 implies that the probability that the “correct” option will beat all other options in pairwise majority voting converges to 1 as the number of individuals increases (List and Goodin 2001). Specifically, C1 implies the conditions of proposition 11b, and, in consequence, the probability of a cycle will converge to 0.

Let us turn to the aggregation over multiple propositions. If the premises P and Q are both true, then C2 implies the conditions of proposition 2a, and the probability of a collective inconsistency under propositionwise majority voting converges to 1 as the number of individuals increases (see proposition 3).

These considerations break the apparent similarity between the probability of cycles and the probability of inconsistent collective sets of judgments. In short, C1 implies the conditions of proposition 11b, whereas C2 implies the conditions of proposition 2a. If individuals have a level of competence that is better than random but not especially high, then the probability of a Condorcet paradox will converge to 0 while the probability of a doctrinal paradox will converge to 1. Given the results of section 4, we may expect this effect to be even greater when the number k of premises is large. If there are k premises (supposing, for our argument, all are true), any level of individual competence above 0.5 but below $\sqrt[k]{(0.5)}$ implies that the probability of inconsistent collective judgments converges to 1 as the number of individuals increases.

The predicted discrepancy between the probability of cycles and the probability of inconsistent collective sets of judgments seems consistent with two pieces of anecdotal evidence. The predicted low probability of cycles in a large electorate (so long as we are not in an impartial culture) seems consistent with the striking lack of empirical evidence for cycles (see Mackie 2000 for a critique of several purported empirical examples of cycles). The predicted higher probability of doctrinal paradoxes in a large electorate (even when we are not in an impartial culture) seems consistent with the findings of an empirical study of voting on referenda (Brams, Kilgour and Zwicker 1997). The study showed that, for three related propositions on the environment in a 1990 referendum in California, less than 6% of the (sampled) electorate individually endorsed the particular conjunction of these three propositions (acceptance of two, rejection of the third) that won under propositionwise majority voting. If the winning combination of propositions were to serve as jointly necessary and sufficient premises for some other conclusion or if a separate vote had been taken on the particular winning conjunction (which would presumably fail to get majority support), we would have a straightforward instance of an inconsistent collective set of judgments.

6 Conclusion

The aim of this paper has been to discuss the likelihood of collective inconsistencies under propositionwise majority voting. We have developed a model for determining the probability of such inconsistencies, and applied the model to conjunctive and disjunctive versions of the doctrinal paradox with two premises, and also to the conjunctive version of the paradox with more than two premises.

We have identified conditions under which the probability of collective inconsistencies under propositionwise majority voting converges to 1 and conditions under which it converges to 0. Both sets of conditions can occur in plausible circumstances. In the case of the conjunctive version of the doctrinal paradox, convergence of the probability of the paradox to 1 is implied by standard competence assumptions in a Condorcet jury framework when all premises are true and individual competence is not particularly high. Convergence of the probability of the paradox to 0 occurs when either at least one of the premises is false or individual competence is very high. In the disjunctive case, convergence of the probability of the paradox to 1 occurs when all premises are false and individual competence is not particularly high. Convergence of the probability of the paradox to 0 occurs when either at least one of the premises is true or individual competence is very high.

Since decision problems with medium individual competence seem empirically plausible, the occurrence of the doctrinal paradox may be quite likely. This reinforces the importance of identifying escape-routes from the paradox and of asking what methods groups can and do employ to avoid the paradox (see also List and Pettit 2002a).

With regard to possible escape-routes, following Bovens and Rabinowicz (2001a, 2001b), we have seen that, for a large class of cases, the premise-based procedure of decision-making is superior to the conclusion-based procedure in terms of tracking the "truth" (where there is a truth to be tracked), especially when we are concerned with tracking the "truth" for the right reasons. This suggests a happy coincidence between epistemic and procedural perspectives on the two alternative decision procedures. While the arguments offered by Pettit (2001) and Chapman (2002) can be interpreted as procedural arguments in favour of the premise-based procedure, we have here seen that, in a large class of cases, the premise-based procedure will also be preferred on epistemic grounds. Finally, we have compared the present results with existing results on the probability of Condorcet's paradox.

The present results should be viewed as initial results, not as the final word, on the probability of collective inconsistencies under propositionwise majority voting. More sophisticated probability-theoretic models could be constructed, for instance allowing different probabilities corresponding to different individuals, and certain dependencies between the judgments of different individuals or between the same individual's judgments on different propositions (compare the discussion at the end of section 3). But even the present initial results support one conclusion. The occurrence of the doctrinal paradox is not implausible at all, and the paradox deserves attention.

Appendix 1: Calculating the Probability of a Collective Inconsistency under Propositionwise Majority Voting for Finite Values of *n*

Let X_{PQ}, $X_{P\neg Q}$, $X_{\neg PQ}$, $X_{\neg P\neg Q}$ be the random variables whose values are the numbers of individuals holding the sets of judgments PQ, $P\neg Q$, $\neg PQ$, $\neg P\neg Q$, respectively. The joint distribution of X_{PQ}, $X_{P\neg Q}$, $X_{\neg PQ}$, $X_{\neg P\neg Q}$ is a multinomial distribution with the following probability function:

$$P(X_{PQ}=n_{PQ}, X_{P\neg Q}=n_{P\neg Q}, X_{\neg PQ}=n_{\neg PQ}, X_{\neg P\neg Q}=n_{\neg P\neg Q})$$

$$= \frac{n!}{n_{PQ}!\; n_{P\neg Q}!\; n_{\neg PQ}!\; n_{\neg P\neg Q}!}\, p_{PQ}{}^{n_{PQ}}\, p_{P\neg Q}{}^{n_{P\neg Q}}\, p_{\neg PQ}{}^{n_{\neg PQ}}\, p_{\neg P\neg Q}{}^{n_{\neg P\neg Q}}\,.$$

Using proposition 1 and the stated probability function, we can infer the following proposition on the probability of collective inconsistencies under propositionwise majority voting.

Proposition 12. *Let the connection rule be* $(R \leftrightarrow (P \wedge Q))$. *Suppose there are n individuals, where each individual has independent probabilities* p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$ *of holding the sets of judgments* PQ, $P\neg Q$, $\neg PQ$, $\neg P\neg Q$, *respectively. Then the probability that there will be a collective inconsistency under propositionwise majority voting is*

$$P((X_{PQ}+X_{P\neg Q}> n/2) \textit{ and } (X_{PQ}+X_{\neg PQ}> n/2) \textit{ and } (X_{PQ}< n/2))$$

$$= \sum_{<n_{PQ},n_{P\neg Q},n_{P\neg Q},n_{\neg P\neg Q}>\in N_{PQ\neg R}} \frac{n!}{n_{PQ}!\; n_{P\neg Q}!\; n_{\neg PQ}!\; n_{\neg P\neg Q}!}\, p_{PQ}{}^{n_{PQ}}\, p_{P\neg Q}{}^{n_{P\neg Q}}\, p_{\neg PQ}{}^{n_{\neg PQ}}\, p_{\neg P\neg Q}{}^{n_{\neg P\neg Q}},$$

where $N_{PQ\neg R}$:= $\{<n_{PQ}, n_{P\neg Q}, n_{\neg PQ}, n_{\neg P\neg Q}> : (n_{PQ} + n_{P\neg Q} > n/2)$ *and* $(n_{PQ} + n_{\neg PQ} > n/2)$ *and* $(n_{PQ} < n/2)$ *and* $(n_{PQ} + n_{P\neg Q} + n_{\neg PQ} + n_{\neg P\neg Q} = n)\}$ (*set of all vectors* $<n_{PQ}, n_{P\neg Q}, n_{\neg PQ}, n_{\neg P\neg Q}>$ *for which there are majorities for each of P and Q, and a majority against R*).

The probabilities of all other logically possible combinations of majorities for or against *P*, *Q* and *R* can be calculated analogously.

Appendix 2: Calculating the Probability of the Various Scenarios in Table 5

For each of the 10 scenarios in table 5, let *M* be the set of all vectors $<n_{PQ}, n_{P\neg Q}, n_{\neg PQ}, n_{\neg P\neg Q}>$ (with sum *n*) for which the condition corresponding to the relevant scenario is satisfied. Using the probability function for the joint distribution of X_{PQ}, $X_{P\neg Q}$, $X_{\neg PQ}$, $X_{\neg P\neg Q}$ (see appendix 1), the desired probability is

$$\sum_{<n_{PQ},n_{P\neg Q},n_{\neg PQ},n_{\neg P\neg Q}>\in M} \frac{n!}{n_{PQ}!\, n_{P\neg Q}!\, n_{\neg PQ}!\, n_{\neg P\neg Q}!} p_{PQ}^{\,n_{PQ}}\, p_{P\neg Q}^{\,n_{P\neg Q}}\, p_{\neg PQ}^{\,n_{\neg PQ}}\, p_{\neg P\neg Q}^{\,n_{\neg P\neg Q}}.$$

For example, if *P* and *Q* are both false and we are interested in the probability that the conclusion-based procedure reaches the correct decision on *R* for the right reasons (scenario 10), then we simply put $M := \{<n_{PQ}, n_{P\neg Q}, n_{\neg PQ}, n_{\neg P\neg Q}> : (n_{\neg P\neg Q} > n/2)$ *and* $(n_{PQ} + n_{P\neg Q} + n_{\neg PQ} + n_{\neg P\neg Q} = n)\}$.

Appendix 3: Proofs

A *condition* ϕ *on a set of k probabilities*, $p_1, p_2, \ldots, p_k$, is a mapping whose domain is the set of all logically possible assignment of probabilities to $p_1, p_2, \ldots, p_k$ and whose co-domain is the set {*true*, *false*}. Whenever $\phi(p_1, p_2, \ldots, p_k) =$ *true*, we shall say that the probabilities $p_1, p_2, \ldots, p_k$ satisfy ϕ; and whenever $\phi(p_1, p_2, \ldots, p_k) =$ *false*, we shall say the probabilities $p_1, p_2, \ldots, p_k$ violate ϕ.

Examples of ϕ for the probabilities $p_{PQ}, p_{P\neg Q}, p_{\neg PQ}, p_{\neg P\neg Q}$ are

- $(p_{PQ} + p_{P\neg Q} > 1/2)$ *and* $(p_{PQ} + p_{\neg PQ} > 1/2)$ *and* $(p_{PQ} < 1/2)$
- $(p_{PQ} \geq 1/2)$
- $(p_{PQ} > 1/2)$ *and* $(p_{P\neg Q} > 1/2)$

A condition ϕ is *consistent* if there exists at least one logically possible assignment of probabilities to $p_1, p_2, \ldots, p_k$ satisfying ϕ. A condition ϕ is *strict* if, for every assignment of probabilities $p_1, p_2, \ldots, p_k$ satisfying ϕ, there exists an $\varepsilon > 0$ such that, whenever the probabilities $p^*_1, p^*_2, \ldots, p^*_k$ lie inside a sphere in $\mathbf{R}^k$ with centre $p_1, p_2, \ldots, p_k$ and radius ε, then the probabilities $p^*_1, p^*_2, \ldots, p^*_k$ also satisfy ϕ. It is easily seen that the condition ($p_{PQ} + p_{P\neg Q} > 1/2$) *and* ($p_{PQ} + p_{\neg PQ} > 1/2$) *and* ($p_{PQ} < 1/2$) is both consistent and strict; the condition ($p_{PQ} \geq 1/2$) is consistent, but not strict; and the condition ($p_{PQ} > 1/2$) *and* ($p_{P\neg Q} > 1/2$) is not consistent.

Let $X_1, X_2, \ldots, X_k$ be a set of k random variables whose joint distribution is a multinomial distribution with the following probability function:

$$P(X_1{=}n_1, X_2{=}n_2, \ldots, X_k{=}n_k) = \frac{n!}{n_1!\, n_2! \ldots n_k!} p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k},$$

where $n_1 + n_2 + \ldots + n_k = n$.

Lemma 1 (Convergence Lemma). *Let ϕ be any consistent strict condition on a set of k probabilities, and suppose the probabilities $p_1, p_2, \ldots, p_k$ satisfy ϕ. Then $P(X_1/n, X_2/n, \ldots, X_k/n$ satisfy $\phi)$ converges to 1 as n tends to infinity.*

Proof of lemma 1. Consider the vector of random variables $\underline{X}^* = \langle X^*_1, X^*_2, \ldots, X^*_k \rangle$, where, for each i, $X^*_i := X_i/n$. We know that the joint distribution of $n\underline{X}^*$ is a multinomial distribution with mean vector $n\underline{p} = \langle np_1, np_2, \ldots, np_k \rangle$ and with variance-covariance matrix $n\Sigma = (s_{ij})$, where, for each i, j, $s_{ij} = np_i(1-p_i)$ if $i=j$ and $s_{ij} = -np_ip_j$ if $i \neq j$. By the central limit theorem, for large n, $(\underline{X}^*-\underline{p})\sqrt{(n)}$ has an approximate multivariate normal distribution $N(\underline{0}, \Sigma)$, and $\underline{X}^*-\underline{p}$ has an approximate multivariate normal distribution $N(\underline{0}, {}^1/_n\, \Sigma)$. Let $f_n : \mathbf{R}^k \rightarrow \mathbf{R}$ be the corresponding density function for $\underline{X}^*-\underline{p}$. Using this density function, $P(X_1/n, X_2/n, \ldots, X_k/n$ *satisfy* $\phi) \approx \int_{\underline{t} \in S} f_n(\underline{t})d\underline{t}$, where

$$S := \{\underline{t} = \langle t_1, t_2, \ldots, t_k \rangle \in \mathbf{R}^k : (t_1+p_1), (t_2+p_2), \ldots, (t_k+p_k) \textit{ satisfy } \phi\}.$$

By assumption, the probabilities $p_1, p_2, \ldots, p_k$ satisfy ϕ, and hence $\underline{0} \in S$. Since ϕ is strict, there exists an $\varepsilon > 0$ such that $S_{\underline{0},\varepsilon} \subseteq S$, where $S_{\underline{0},\varepsilon}$ is a sphere in $\mathbf{R}^k$ around $\underline{0}$ with radius ε. Then, since f_n is nonnegative, $\int_{\underline{t} \in S} f_n(\underline{t})d\underline{t} \geq \int_{\underline{t} \in S_{\underline{0},\varepsilon}} f_n(\underline{t})d\underline{t}$. But, as f_n is the density function corresponding to $N(\underline{0}, {}^1/_n\, \Sigma)$, $\int_{\underline{t} \in S_{\underline{0},\varepsilon}} f_n(\underline{t})d\underline{t} \rightarrow 1$ as $n \rightarrow \infty$, and hence $\int_{\underline{t} \in S} f_n(\underline{t})d\underline{t} \rightarrow 1$ as $n \rightarrow \infty$, as required.

Proof of proposition 2.

(a) ($p_{PQ} + p_{P\neg Q} > 1/2$) *and* ($p_{PQ} + p_{\neg PQ} > 1/2$) *and* ($p_{PQ} < 1/2$) is a consistent strict condition. By lemma 1, $P((X_{PQ} + X_{P\neg Q} > n/2)$ *and* $(X_{PQ} + X_{\neg PQ} > n/2)$ *and* $(X_{PQ} < n/2)) \to 1$ as $n \to \infty$. The result then follows from proposition 1.

(b) ($p_{PQ} + p_{P\neg Q} < 1/2$) *or* ($p_{PQ} + p_{\neg PQ} < 1/2$) *or* ($p_{PQ} > 1/2$) is a consistent strict condition. By lemma 1, $P((X_{PQ} + X_{P\neg Q} < n/2)$ *or* $(X_{PQ} + X_{\neg PQ} < n/2)$ *or* $(X_{PQ} > n/2)) \to 1$ as $n \to \infty$. The result then follows from proposition 1.

Proof of proposition 3.

(a) P and Q are true.

The relevant case in table 4 is case 1. For the first part, it is sufficient to show that p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$ satisfy the conditions of proposition 2a. Suppose $0.5 < p, q < \sqrt{(0.5)}$. Then

$$p_{PQ} + p_{P\neg Q} = pq + p(1\text{-}q) = p > 0.5$$
$$p_{PQ} + p_{\neg PQ} = pq + (1\text{-}p)q = q > 0.5$$
$$p_{PQ} = pq < 0.5,$$

as required. For the second part, it is sufficient to show that p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$ satisfy the conditions of proposition 2b. Suppose $\sqrt{(0.5)} < p, q$. Then

$$p_{PQ} = pq > 0.5,$$

as required.

(b) Not both P and Q are true.

The relevant cases in table 4 are cases 2, 3 and 4. It is sufficient to show that p_{PQ}, $p_{P\neg Q}$, $p_{\neg PQ}$, $p_{\neg P\neg Q}$ satisfy the conditions of proposition 2b. Suppose $0.5 < p, q$.

In case 2, $p_{PQ} + p_{\neg PQ} = p(1\text{-}q) + (1\text{-}p)(1\text{-}q) = 1\text{-}q < 1/2$, as required.

In case 3, $p_{PQ} + p_{P\neg Q} = (1\text{-}p)q + (1\text{-}p)(1\text{-}q) = 1\text{-}p < 1/2$, as required.

In case 4, $p_{PQ} + p_{P\neg Q} = (1\text{-}p)(1\text{-}q) + (1\text{-}p)q = 1\text{-}p < 1/2$, as required.

Proof of proposition 4.

(a) Suppose $0.5 < p, q$. It is sufficient to show that the probability that the premise-based procedure reaches a correct decision on R for the right reasons (implying also that it reaches a correct decision regardless of reasons) converges to 1 as n tends to infinity. Consider the four cases in table 4.

Case 1: $p_{PQ} + p_{P\neg Q} = pq + p(1\text{-}q) = p > 0.5$ *and* $p_{PQ} + p_{\neg PQ} = pq + (1\text{-}p)q = q > 0.5$, a consistent strict condition. By lemma 1, $P((X_{PQ} + X_{P\neg Q} > n/2)$ *and* $(X_{PQ} + X_{\neg PQ} > n/2)) \to 1$ as $n \to \infty$ (compare condition (1) in table 5).

All other cases are analogous. In each case, the relevant consistent strict condition will be identified, and the result will follow from lemma 1.

Case 2: $p_{PQ}+p_{P\neg Q} = p(1\text{-}q) + pq = p > 0.5$ *and* $p_{P\neg Q} + p_{\neg P\neg Q} = pq + (1\text{-}p)q = q > 0.5$. $P((X_{PQ} + X_{P\neg Q} > n/2)$ *and* $(X_{P\neg Q} + X_{\neg P\neg Q} > n/2)) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (4) in table 5).

Case 3: $p_{\neg PQ} + p_{\neg P\neg Q} = pq + p(1\text{-}q) = p > 0.5$ *and* $p_{PQ} + p_{\neg PQ} = (1\text{-}p)q+pq = q > 0.5$. $P((X_{\neg PQ} + X_{\neg P\neg Q} > n/2)$ *and* $(X_{PQ} + X_{\neg PQ} > n/2)) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (5) in table 5).

Case 4: $p_{\neg PQ}+p_{\neg PQ} = p(1\text{-}q) + pq = p > 0.5$ *and* $p_{P\neg Q}+p_{\neg P\neg Q} = (1\text{-}p)q+pq = q > 0.5$. $P((X_{\neg PQ} + X_{\neg PQ} > n/2)$ *and* $(X_{P\neg Q} + X_{\neg P\neg Q} > n/2)) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (6) in table 5).

(b) Suppose $0.5 < p, q < \sqrt{(0.5)}$, and both P and Q (and by implication R) are true. Then $p_{PQ} = pq < 0.5$. $P(X_{PQ} < n/2) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (2) in table 5).

(c) Suppose $0.5 < p, q < \sqrt{(0.5)}$, and not both P and Q are true. By part (a) (cases 2, 3 and 4), the probability that there will not be a majority for P and a majority for Q converges to 1 as n tends to infinity. This implies in particular that $P(X_{PQ} < n/2) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (7) in table 5).

(d) Suppose $0.5 < p, q < \sqrt{(0.5)}$. The relevant cases in table 4 are cases 2, 3 and 4.

Case 2: $p_{P\neg Q} = pq < 0.5$. $P(X_{P\neg Q} < n/2) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (8) in table 5).

Case 3: $p_{\neg PQ} = pq < 0.5$. $P(X_{\neg PQ} < n/2) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (9) in table 5).

Case 4: $p_{\neg P\neg Q} = pq < 0.5$. $P(X_{\neg P\neg Q} < n/2) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (10) in table 5).

(e) Suppose $p, q > \sqrt{(0.5)}$. It is sufficient to show that the probability that the conclusion-based procedure reaches a correct decision on R for the right reasons (implying also that it reaches a correct decision regardless of reasons) converges to 1 as n tends to infinity. Consider the four cases in table 4.

Case 1: $p_{PQ} = pq > 0.5$. $P(X_{PQ} > n/2) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (2) in table 5).

Case 2: $p_{P\neg Q} = pq > 0.5$. $P(X_{P\neg Q} > n/2) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (8) in table 5).

Case 3: $p_{\neg PQ} = pq > 0.5$. $P(X_{\neg PQ} > n/2) \rightarrow 1$ as $n \rightarrow \infty$ (compare condition (9) in table 5).

Case 4: $p_{\neg P\neg Q} = pq > 0.5$. $P(X_{\neg P\neg Q} > n/2) \rightarrow 1$ as $n \rightarrow \infty$

(compare condition (10) in table 5).

Proof of proposition 8.
Given unanimous acceptance of $(S \leftrightarrow (P \wedge Q \wedge R))$, the conditions of proposition 8a are necessary and sufficient for the majority acceptance of the (inconsistent) set of propositions $\{P, Q, R, (S \leftrightarrow (P \wedge Q \wedge R)), \neg S\}$. To prove propositions 8b and 8c, it is sufficient to note that

$(p_{PQR}+p_{PQ\neg R}+p_{P\neg QR},+p_{P\neg Q\neg R} > 1/2)$ *and* $(p_{PQR}+p_{PQ\neg R}+p_{\neg PQR}+p_{\neg PQ\neg R} > 1/2)$ *and* $(p_{PQR}+p_{P\neg QR}+p_{\neg PQR}+p_{\neg P\neg QR} > 1/2)$ *and* $(p_{PQR} < 1/2)$

and

$(p_{PQR}+p_{PQ\neg R}+p_{P\neg QR},+p_{P\neg Q\neg R} < 1/2)$ *or* $(p_{PQR}+p_{PQ\neg R}+p_{\neg PQR}+p_{\neg PQ\neg R} < 1/2)$ *or* $(p_{PQR}+p_{P\neg QR}+p_{\neg PQR}+p_{\neg P\neg QR} < 1/2)$ *or* $(p_{PQR} > 1/2)$

are each consistent strict conditions. The desired results then follow from lemma 1 and proposition 8a.

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