

# Cancellation and Uncertainty Aversion on Limit Order Books

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## Abstract

This paper models limit order books where each trader is uncertain of the underlying distribution in the asset's value to others. If this uncertainty is rapidly resolved, *fleeting* limit orders are submitted and quickly cancelled. This enhances liquidity supply, but leaves intact established comparative statics results on spreads. However, risk neutral liquidity suppliers are averse to persistent uncertainty due to concavity in the function describing limit order utility, and spreads widen. This helps explain wide spreads in the morning.

The model describes traders who in equilibrium correctly anticipate market orders' endogenous stochastic intensities. It highlights how limit orders queue for execution.

*JEL classification:* D8, G1

*Keywords:* market microstructure, limit order book, fleeting orders, order cancellation.

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# 1 Introduction

According to Hasbrouck and Saar (2002) as many as 27.7 per cent of all offers to trade submitted on the Island ECN exchange for NASDAQ securities are cancelled within two seconds of being submitted.<sup>1</sup> What is the benefit of placing these “fleeting” limit orders for such a brief time? Do they have a significant impact on the bid-ask spread or on its sensitivity to trader impatience or numbers?

Theoretical explanations are available for the stylized fact that in limit order markets bid-ask spreads enlarge at the end of the trading day.<sup>2</sup> Yet on the London Stock Exchange SETS platform, these tend to be considerably smaller than the wide spreads at the start of the day. Why are spreads not bid down in the morning by patient placers of limit orders who have the whole day to transact?

This paper explains both these phenomena within a single modelling framework, whose central notion is uncertainty on the part of traders about the attributes of other traders in the market. Remarkably, risk neutral traders are averse to offering liquidity when faced with uncertainty of this sort, and spreads must widen to clear the market. This helps explain wide spreads during morning trading, since in the morning traders are particularly uncertain who else is operating in the market. On the other hand, uncertainty can also *encourage* the submission of limit orders, since the option to cancel if bad news arrives restricts their downside risk, while their upside remains. Since bad news does sometimes arrive, this can imply fleeting order placement.

These ideas are developed in the context of an electronic limit order book, the trading system now used by many international security markets, among them the London Stock Exchange. To trade, market participants accept outstanding offers to buy or sell (collected in the *book*), or place offers of their own. If a participant accepts an outstanding offer (by making a *market order*) she avoids delay but *ex ante* placing a new offer (a *limit order*) in the book improves her execution price.<sup>3</sup>

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<sup>1</sup>Hasbrouck and Saar (2002) study trading on the Island ECN exchange in the stocks of the 300 largest firms in the NASDAQ National Market (ranked by equity capitalization) over the fourth quarter of 1999.

<sup>2</sup>This phenomenon is noted in McNish and Wood (1992). An explanation in terms of trader impatience and exposure to execution risk is offered in Foucault (1999).

<sup>3</sup>During the normal trading day, traders on limit order books participate in a continuous time auction. Rather than trading directly with each other, they place limit orders in the book. An order specifies a limit price drawn from a lattice with constant price intervals, a quantity, and a trade direction. It is a commitment to trade up to the specified quantity of the asset at prices at least as favorable as the

The paper proposes that when a trader (or her computer system<sup>4</sup>) places a fleeting order, it is as if she took out an option, while not cancelling is like exercising the option. For some traders it would be negligent not to place a limit order – just to check for impatient trading partners who would accept its price. However, the order should be left in place only so long as this upside risk remains. Once it becomes apparent that trading partners arrive at the market with a low frequency, it is preferable to escape the limit order’s execution risk and delay by cancelling it. If this becomes apparent quickly, cancellation is rapid and limit orders are *fleeting*. Fleeting orders have only an ephemeral effect on the order book, making the spread a little tighter but leaving unchanged its comparative statics with respect to trader impatience and trading volumes.

Yet this result is turned on its head when traders face an uncertainty of a sort that persists. Then, limit orders are stripped of their option value, but remain subject to uncertainty about their expected time to execution. Due to a concavity in the function describing limit order utility, the uncertainty deters liquidity supply: it deters the submission of limit orders since traders place high weight on the downside risk of waiting a very long time for them to execute. In empirical market microstructure work, a wide spread is often used as evidence of informed trading.<sup>5</sup> The results of this paper suggest that this role as a proxy may need to be reviewed.

To describe market uncertainty in a closed model, this paper innovates in two significant areas. First, it models traders who in equilibrium correctly anticipate the endogenous stochastic arrival rates of orders, and correctly infer from this the expected risk and delay of placing a limit order. This can be contrasted with Domowitz and Wang (1981), where the arrival rates are simply exogenous parameters, or Hollifield, Miller, and Sandas

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specified limit price. The order book translates these commitments into trades using a queuing rule that follows strict price, then time priority. Orders may be executed in one or more trades. Orders can be cancelled at any time.

<sup>4</sup>Fleeting orders are often placed by computer algorithms managed by traders. This permits traders to implement a more complex order placement process than would be feasible by hand. This paper treats such algorithms just as it treats traders’ judgement: as a way of producing an optimal response to market conditions.

<sup>5</sup>For example Engle and Russell (1998) use an Autoregressive Condition Duration model to estimate order dynamics for IBM stock using the TORQ data set. Different dynamics are observed during periods of wide spreads than during periods of narrow spreads. This distinction is used to break up information-based effects from effects due to liquidity clustering. Hasbrouck (1991), although cautious about interpreting spreads as evidence of informed trading, uses a Vector Autoregression (VAR) to estimate spread dynamics that are consistent with a market maker who infers from a large trade an increased likelihood that an information event has occurred. See also Chiang and Venkatesh (1988).

(2003)<sup>6</sup>, where the risk that a limit order does not execute is anticipated by the trader who places it conditional on the state of the limit order book but this anticipation is not shown to be based in a rational assessment of order arrival rates due to other traders' actions.

Second, the model relaxes a common, but restrictive, assumption in a way which is novel (so far as I know) in the market microstructure literature. It does not rely on the assumption that the distribution of trader reservation values for the traded asset (hereafter their *reservation values*) is common knowledge. On the contrary, it treats traders' lack of knowledge as central. By contrast, much recent research in this area, for example Parlour (1998), Hollifield, Miller, and Sandas (2003) and Foucault, Kadan, and Kandel (2003), has made use of models where trader reservation values are independently drawn from a common, and commonly known, distribution. Such frameworks tend to deliver important results about optimal order placement and the evolution of bid-ask spreads or order book depths.

But the common knowledge assumption does not allow for the possibility that traders might learn useful information about a market's composition by watching it, and it rules out a desirable feature in models of insider trading. This paper examines the effect on traders of uncertainty about the number and distribution of trader types. It develops a model where rather than the distribution of trader types, say  $F$ , or their number, say  $N$ , being common knowledge, instead traders share a common prior over the set of possible states of the market,  $\Theta$ . Each state of the market,  $\varphi \in \Theta$ , implies a pair  $(N_\varphi, F_\varphi)$ .

Together, these two innovations imply a model where the trader submitting a limit order is uncertain of the frequency of the market orders that trigger its execution, for even though she anticipates it correctly for any given  $\varphi$ , she is uncertain of  $\varphi$  itself.

On markets with many participants, the expected utility from limit orders is a concave function of liquidity demand. More precisely, it is increasing but concave (except near zero) in the anticipated frequency of opposing market orders. It is increasing because on average a limit order is executed more quickly and surely if opposing market orders arrive faster. Intuitively, it is concave except near zero because it is an increasing function which is bounded above.<sup>7</sup> It is bounded above by the payoff were it to be executed

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<sup>6</sup>See also Hollifield, Miller, Sandas, and Slive (2003).

<sup>7</sup>Strictly, it is not necessarily the case that there exists some threshold above which an increasing function which is bounded above is concave. Consider for example the function  $x \rightarrow ((\sin x + x)^{-1} + 1)^{-1}$ .

immediately - i.e. the difference between price and reservation value (see Figure 2 which describes a limit bid at price  $B$ ). Thus, a typical limit bidder might be largely indifferent if market sales arrive at 1,000 per minute or 2,000 per minute, but might care more whether they arrive at one per minute or two per minute. Uncertainty on the part of risk neutral traders about trader numbers or the distribution from which reservation values are drawn implies uncertainty about the arrival rates of market orders. Due to the concavity discussed above, persistent uncertainty deters limit orders and causes the spreads that equalize the supply and demand for liquidity to widen. In short, limit order placers behave as if they were “uncertainty averse”.

But the picture is different in the plausible case that traders may learn about the composition of the market by participating in it. Due to the optionality provided by cancellation, they face a kink in the concave function described above making it locally convex (depicted in Figure 4). Rather than exiting the market immediately by placing a market order, some traders place limit orders knowing that they may prefer to cancel when news arrives. This is typical of limit order markets: traders bear the delay and risk of submitting limit orders to test for a weight of trading in the opposite direction. But such a weight of trading may be ruled out quickly when it does not happen. If this is so, then cancellation, where it occurs, is rapid: there are *fleeting* limit orders.

On the other hand, some traders do not cancel, who would never have placed a limit order had they known more about the market at the outset. They are “coaxed” into supplying additional liquidity. The uncertainty can be thought of as causing these traders to act as if the order book depth were less than it truly is. Despite the fact that the bid-ask spread must narrow somewhat to offset this surplus in liquidity supply, the existence of fleeting orders leaves intact established results concerning the effects on equilibrium spreads of trader impatience noted in Foucault (1999) or of trader numbers.

The paper proceeds as follows. Section 2 introduces a basic model in which the distribution of reservation values is common knowledge. Section 3 characterizes equilibrium. Section 4 solves the basic model in a natural symmetric case to which uncertain conditions may be compared. Section 5 describes the concave functional form that is critical to the result of Section 6, that spreads widen under persistent uncertainty. This is matched in Section 7 by a complementary result, that where traders rapidly learn about a market by participating in it, fleeting orders are placed. Section 8 provides a broad discussion

of the model. Section 9 concludes.

## 2 The basic model

The basic model is of considerable interest in itself. Its purpose is to serve as a benchmark in which traders are *not* uncertain about the distribution and number of trader types. It is innovative in that in equilibrium orders arrive according to Poisson processes whose intensities are correctly anticipated by traders, conditional on the best prices in the limit order book and its depth. It shares with Hollifield, Miller, Sandas, and Slive (2003) the feature that these Poisson processes arise from heterogenous traders' randomly timed participation in the market. It follows Parlour (1998) (which develops a discrete time model) in making a stylized assumption of only two admissible prices, but in the current model these prices are determined in equilibrium by a market clearing condition.

Traders on the London Stock Exchange report that they would typically aim to “pay no more than half the spread” over the course of a fragmented order placement. Thus, they would aim to place at least half its volume through limit rather than market orders. However, placing limit orders encounters risk and delay. The basic model reflects this. In it, risk neutral market participants trade off the cost of immediate order execution against the cost of delayed execution. Immediate execution is performed at a disadvantageous price. Delayed execution, on the other hand, is unattractive to impatient traders. The cost of delayed execution is increasing in the margin between traders' reservation values and the price as well as in their impatience, but the cost of immediate execution is insensitive to either. Therefore, traders with values drawn near to their median value submit limit orders. Conversely, traders with extreme high or low values submit market orders. The basic model therefore shows how a homogeneous population of traders, differing only in their private values for the asset, can organize themselves into liquidity providers (dealers who submit limit orders) and price takers (investors who submit market orders).

The model is closed in two steps. First (*rational expectations*), traders are assumed to anticipate correctly the market and limit order arrival rates at the bid and ask queues. They condition their choice of order on these rates. Second (*market clearing in expectation*), prices move so that the supply and demand for liquidity are equal in expectation. More precisely, prices move so that the equilibrium intensity of limit bids (asks) is equal

to the opposing intensity of market sales (purchases).

In the model, the timing of traders' actions is stochastic, they set prices competitively, and they have no budget or short-selling constraint. In addition, the model does not explain the determinants of order book depth. Instead, depth is treated as fixed. Discussion of these assumptions, which together permit parsimonious analysis of the central theme, uncertainty, is provided in Section 8.

## 2.1 Trader preferences

A large number,  $N$ , of risk neutral traders independently draw private reservation values for the traded asset,  $V$ , from a distribution on  $\mathbb{R}^+$  with continuous cumulative density function (CDF)  $F$ . All traders discount future payoffs at rate  $\rho > 0$ . The value  $\rho$  quantifies the traders' preference for trading earlier rather than later. This preference may be due to impatience, or to traders believing that later execution may never occur, or a combination. On the other hand,  $V$  determines the importance to the trader of trading at all, whether early or late. It encapsulates idiosyncratic values attached to holding the asset, such as those due to endowment shocks.

## 2.2 Market Structure

Time is continuous and runs on indefinitely from 0, when types are assigned. Traders face only two prices: a bid price  $B$  and an ask price  $A$ , where  $A > B > 0$ . These prices are constant, but are endogenous to the model in equilibrium. Outstanding offers to trade are all at these prices: offers to buy (limit bids) at price  $B$  and offers to sell (limit asks) at price  $A$ . They are executed in the sequence in which they were submitted, hence they form two queues. The total volume offered in the bid (ask) queue is denoted  $L_B$  ( $L_A$ ). All orders and transactions in this model are of unit quantity, therefore  $L_B$  and  $L_A$  are equivalently the number of orders queueing at these two prices: that is, the lengths of the two queues. This paper is not concerned with the determinants of the parameters  $L_B$  and  $L_A$ , which depend importantly on the minimum admissible price increment on the limit order book. They are therefore treated as exogenous and fixed.

Each trader has a countably infinite series of opportunities to place a limit or market order for the traded asset. These *order opportunities* arrive randomly according to a Poisson process of parameter  $I$ . With probability  $I\delta t$  each trader has an order opportu-

nity in any small period of time  $(t, t + \delta t)$ . Throughout this paper, parameters such as  $I$  are referred to as the *intensities* of the related Poisson process.

At each order opportunity, traders choose one of four actions or no action at all. Each action is to place an order of unit volume. The actions are to place a buy limit order (limit bid), a buy market order, a sell limit order (limit ask), or a sell market order. From a buy market order a trader receives an immediate payoff of  $(V - A)$ , while the payoff to sell market orders is  $(B - V)$ . A limit order may be cancelled and immediately replaced by any other order at any time. If a limit bidder does not cancel, she receives  $(V - B)$  at the time of the  $(L_B + 1)$ th subsequent market sale. An equivalent rule holds for asks.

This can be interpreted in the following way. To place a market purchase is to execute against the limit ask at the top of the ask queue at price  $A$ . A trader who places a limit bid joins the back of the bid queue of length  $L_B$  at price  $B$ . Analogous interpretations hold for limit asks and market sales.

### 2.3 Incomplete trader information

This part introduces the traders' information structure in some detail to explain how the basic model is later extended to incorporate incomplete information. Traders know their own reservation value,  $V$ . They share a common prior,  $\mathbf{P}$ ,<sup>8</sup> over the set  $\Theta$  of possible states of the market,  $\varphi \in \Theta$ . Different states of the market correspond to different numbers of traders, denoted by  $N_\varphi$ , and different CDFs for others' values, denoted  $F_\varphi$ . The random variables,  $F_\varphi(v)$  and  $N_\varphi$ , are assumed to be independent for any  $v > 0$ . Define the traders' unconditional priors over market conditions,  $(N^*, F^*)$ , by

$$\begin{aligned} N^* & : = E_{\mathbf{P}}\{N_\varphi\}, \\ F^*(v) & : = E_{\mathbf{P}}\{F_\varphi(v)\} \end{aligned}$$

for any  $v \geq 0$ . For clarity of results, it is assumed that  $(N, F) = (N^*, F^*)$ . That is, the traders' unconditional prior over the state space  $\Theta$  is equal to the truth.

In Section 6, where it is established that spreads widen under uncertainty, traders are persistently uncertain of  $\varphi \in \Theta$ . In Section 7, which models fleeting orders, traders enter the market uncertain of  $\varphi \in \Theta$  but rapidly discover that the truth is in fact as their unconditional prior,  $(N, F) = (N^*, F^*)$ . However, in the sections preceding these,

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<sup>8</sup>In this paper the event space and probability distribution representing the beliefs of traders are held implicit, summarized by the variable  $\mathbf{P}$ .

a basic model is analyzed where  $\Theta$  is a singleton containing the truth,  $(N, F)$ , to which the traders attach probability one. That is to say,  $N$  and  $F$  are common knowledge. For expositional ease, no reference is made to  $(N^*, F^*)$ ,  $\varphi$  or  $\Theta$  until Section 6 when incomplete information is introduced.

## 2.4 Equilibrium

Trader strategies are Markov and stationary. They determine an action (limit or market order, buy or sell, or nothing at all) as a function of  $V$  and the traders' commonly held information set,

$$\{A; B; L_A; L_B; \mathbf{P}\}.$$

The action is optimal given that all other traders use the same strategy. If the trader has an order opportunity at time  $t$  she follows the strategy. She cancels a limit order in favor of the relevant market order at time  $t$  only if the conditioning information set changes at  $t$  so she prefers the market order to the existing limit order, taking account of the remaining number of market orders needed for its execution.

The prices  $A$  and  $B$  are set so that, for any time interval, and for either side of the book, the expected number of market orders and the expected number of uncanceled opposing limit orders are equal.

## 3 Solution of the basic model

Each trader's equilibrium action is the same at every order opportunity. Equally, cancellation never occurs in equilibrium, for it can only occur when the parameters of the trader's optimization problem change, which happens only when the order moves up in the queue of orders. But, as an order moves up the queue, it becomes increasingly attractive relative to a market order.

For each trader, with probability one, one of the four payoffs available is greater than the others and greater than zero.<sup>9</sup> Hence with probability one at any order opportunity traders strictly prefer to place one of the four types of orders. Thus each trader generates a Poisson process whose events are of one of the four types of orders. Due to the aggregation

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<sup>9</sup>In a setting of asymmetric information it may no longer be the case that all traders can derive positive expected utility from some order type or other. In particular, traders with valuation within the spread, who would in any event not place a market order, might also not wish to place a limit order if its expected payoff conditional on execution were negative.

properties of independent Poisson processes, in aggregate market orders and limit orders arrive at the market according to Poisson processes. Their arrival rates are constant.

Let traders therefore base their action on the belief that the intensity of the Poisson process describing aggregate market sell (buy) orders (whether submitted at a trader's order opportunity or subsequent to a cancellation) is  $\mu_s^*$  ( $\mu_b^*$ ) and that the intensity of the aggregate uncanceled limit bid (ask) order process is  $\lambda_b^*$  ( $\lambda_s^*$ ). Therefore the expected payoff to placing a limit bid is

$$(V - B)E(e^{-\rho T^s}) \quad (1)$$

where  $T^s$  is the time to the  $(L_B + 1)$ th market sale after  $t$ , i.e. the time to the  $(L_B + 1)$ th event of a Poisson process of intensity  $\mu_s^*$ . Define  $T^b$  to be the equivalent variable on the buy side.

### 3.1 Trading deadline

Traders discount the future at rate  $\rho > 0$ . This is as if traders did not discount the future, but faced a common trading deadline,  $D_\rho$ , at a random time distributed as the time to the first event of a Poisson process with intensity  $\rho > 0$ .<sup>10</sup> No further transactions take place after  $D_\rho$ . Thinking of the model in these terms greatly simplifies its analysis. The expected payoff to a limit bid – the same quantity as (1) – can then be written

$$(V - B) \Pr[T^s < D_\rho]. \quad (2)$$

Equally, a limit ask delivers expected utility  $(A - V) \Pr[T^b < D_\rho]$ .

### 3.2 Cost of immediate execution and cost of delayed execution

At order opportunities, traders select the action which they most prefer. Clearly, traders with  $V > A$  prefer to purchase than sell, but must decide between placing a limit bid and a limit ask. This decision can be formulated as selecting the minimum of two costs defined relative to the unattainable benchmark utility  $(V - B)$ , which would be gained by purchasing immediately at price  $B$ . The cost of delayed execution is the trader's margin  $(V - B)$  multiplied by the probability that the delay caused by joining the back of the bid queue causes the trader to miss the deadline. Thus it is

$$(V - B) \Pr[T^s > D_\rho]. \quad (3)$$

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<sup>10</sup>Conditional on not yet having occurred, the deadline has probability  $\rho \delta t$  of occurring in any small period of time  $(t, t + \delta t)$ .

The cost of immediate execution is simply the spread,  $(A - B)$ . If the cost of immediate execution, the spread, is less than the cost of delayed execution, the trader chooses to place a market order, otherwise she places a limit order.

For buyers, the cost of delayed execution is clearly increasing linearly in  $V$  but is equal to zero at  $V = B$ . Therefore there exists a trader type  $V_b^* > B$  who is indifferent between limit bids and market purchases.  $V_b^*$  will be referred to as the marginal price-taking buyer. All types with higher reservation value prefer market purchases to limit bids and place market purchases at every order opportunity. Similarly, there is a marginal price-taking sellers  $V_s^* < A$ , for whom all types with lower reservation values prefer market sales to limit asks and place market sales at every order opportunity. The condition for the marginal price-taking buyer is

$$(V_b^* - B) \Pr[T^s > D_\rho] = A - B, \quad (4)$$

while the condition for the marginal price-taking seller is given by

$$(A - V_s^*) \Pr[T^b > D_\rho] = A - B. \quad (5)$$

There also exists a trader type  $V^*$ , where  $B < V^* < A$ , who is indifferent between limit bids and limit asks.

$$(V^* - B) \Pr[T^s \leq D_\rho] = (A - V^*) \Pr[T^b \leq D_\rho]. \quad (6)$$

**Interpretation** From (4) and (5), it follows that  $V_b^*$  is greater than  $A$ , and  $V_s^*$  is less than  $B$ . Thus there are buyers and sellers who would gain from a market order, but prefer to submit limit orders. In addition, as  $\rho \rightarrow \infty$ ,  $V_b^* \downarrow A$ . Thus for any  $V > A$ , there exists a (perhaps high) impatience level  $\rho$  such that  $V$  prefers to submit market buy orders.  $V_b^*$  increases with  $\mu_s^*$ . If the expected intensity of market sell orders goes up, more buyers prefer to submit limit orders. Symmetric results hold on the other side of the market.

### 3.3 Aggregating trader behavior

Let the true intensities of the aggregate Poisson processes for market orders, whether submitted at a trader's order opportunity or subsequent to a cancellation, be  $(\mu_b, \mu_s)$  and the Poisson processes for aggregate uncanceled limit orders have intensities  $(\lambda_b, \lambda_s)$ .

These can be calculated by exploiting the additive nature of the intensities of independent Poisson processes. As there is no cancellation in the basic model, all market buy opportunities are submitted at the time of an order opportunity or not at all. Therefore the number of traders submitting market buy orders is approximately<sup>11</sup>

$$N[1 - F(V_b^*)] \tag{7}$$

since this is  $N$  multiplied by the integral of the trader distribution over the part of the trader type space that would choose market buy orders. Each such trader contributes an intensity of  $I$ . It follows that

$$\mu_b = NI[1 - F(V_b^*)]. \tag{8}$$

Similar integrals can be defined over the appropriate intervals for  $\mu_s, \lambda_s$  and  $\lambda_b$ . All these intensities depend on queue lengths and prices, as well as traders' beliefs about  $\mu_s^*$  and  $\mu_b^*$ . Figure 1 illustrates this in a symmetric case.

### 3.4 Rational expectations

In equilibrium traders' strategies are optimal conditional on the information set

$$\{A; B; L_A; L_B; F; N\},$$

and the strategies of the other traders. As has been shown, optimal order placement depends on the anticipated intensities of future order events. Therefore the intensities are rationally anticipated conditional on the behavior of the other traders, i.e.

$$\mu_b = \mu_b^*, \mu_s = \mu_s^*, \lambda_b = \lambda_b^*, \lambda_s = \lambda_s^*. \tag{9}$$

### 3.5 Market clearing in expectation

In equilibrium, prices are such that, for any time interval and for either side of the book, the expected number of market orders and the expected number of uncanceled opposing limit orders are equal. The market orders may have been submitted at a trader's order opportunity or subsequent to a cancellation. The prices can be thought of as those which would clear the market in expectation or the prices at which the supply and demand for

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<sup>11</sup>The approximation becomes exact as  $N$  tends to  $\infty$ .

liquidity are equal. Equivalently, they are the prices where the expected length of either queue is constant.<sup>12</sup> To find them, set

$$\mu_b = \lambda_s, \mu_s = \lambda_b. \quad (10)$$

This identifies a single equilibrium that jointly specifies prices and expected time to limit order execution. It is conditional on the order book depths,  $L_A$  and  $L_B$ .

## 4 Benchmark symmetric example

This section solves the basic model in the symmetric case. The equilibrium characterized here, where market conditions are common knowledge to traders, is used as a benchmark against which to compare settings involving trader uncertainty. The example is designed to be symmetric.

The symmetric example's comparative statics are of considerable interest in themselves. They are consistent with current economic theory on electronic limit order books. In particular, in line with Foucault (1999) and Foucault, Kadan, and Kandel (2003) they predict that bid-ask spreads widen with trader impatience. In addition they predict that an increase in trader numbers, or a decrease in order book depth (the total volume logged in the book), perhaps due to the introduction of a finer lattice of admissible order prices, decrease bid-ask spreads.

**Proposition 4.1** *Suppose that the distribution of trader reservation values,  $F$ , is symmetric about its median,  $m$ , and both queue lengths are equal to  $L > 0$ . Then in any symmetric equilibrium,*

$$\frac{A + B}{2} = m, \quad (11)$$

$$A - B = \left[ F^{-1} \left( \frac{3}{4} \right) - F^{-1} \left( \frac{1}{4} \right) \right] \left[ \frac{1 - \Pr[T^s < D_\rho]}{1 + \Pr[T^s < D_\rho]} \right]. \quad (12)$$

*In addition,  $T^s$ , like  $T^b$ , is distributed as the  $(L + 1)$ th event of a Poisson process of intensity  $\frac{NI}{4}$ .*

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<sup>12</sup>If the depletion rate of the queue is equal to the order arrival rate, the queue length will behave as a random walk in event time with jumps of magnitude  $\pm 1$  arriving according to a Poisson process. This article does not consider the implications of this non-stationarity.

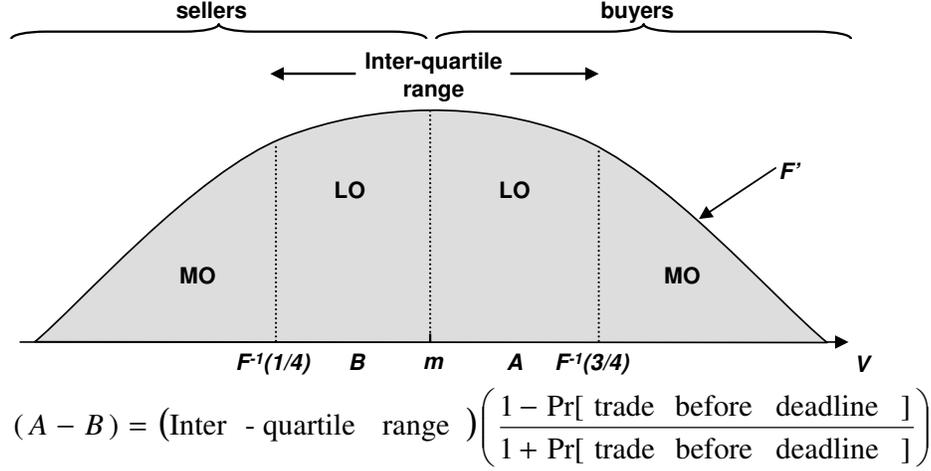


Figure 1: Illustrates the benchmark symmetric equilibrium.

**Proof.** Due to the symmetry of the distribution  $F$ , in a symmetric equilibrium the type space divides into four quartiles representing the four sorts of orders.<sup>13</sup> Hence

$$F(V_b^*) = \frac{3}{4}; F(V_s^*) = \frac{1}{4}; \quad (13)$$

$$\mu_b = \mu_s = \frac{NI}{4}. \quad (14)$$

Therefore, buy and sell market orders both arrive with intensity  $\frac{NI}{4}$ . It then follows that

$$Pr[T^b < D_\rho] = Pr[T^s < D_\rho]. \quad (15)$$

The rest of the proposition is proven by substituting (13) and (15) into (4) and (5), and rearranging. Figure 1 illustrates this symmetric equilibrium. ■

## 4.1 Comparative statics

As a consequence of (12), the spread is a decreasing function of  $Pr[T^s < D_\rho]$  and an increasing function of the inter-quartile range of  $F$ . From this simple observation, a number of comparative static results flow.

**The bid-ask spread,  $A - B$ , increases with impatience,  $\rho$ .** If traders become more impatient, fewer traders prefer limit orders. Spreads must therefore widen to clear the market for liquidity.

<sup>13</sup>For special parameter values asymmetric equilibria are feasible despite the symmetry in  $F$ . These asymmetric equilibria may be ruled out as implausible descriptions of trader behavior.

**The bid-ask spread increases with order book depth,  $L$ .** A deeper book implies that limit orders queue for longer before executing. This makes them less attractive to traders. Spreads must widen to clear the market for liquidity. For example, if pricing rules are adjusted so that admissible prices are less widely spaced (a finer price lattice is used), order queues may tend to be shorter. This would cause the spread to narrow.

**The bid-ask spread tends to zero as  $N \rightarrow \infty$ .** A busier market executes limit orders faster. Hence the cost of delayed execution is low. To clear the market for liquidity, the spread, which fixes the cost of immediate execution, also falls. This result can be interpreted to mean that increased trading volumes reduce the spread.

**The bid-ask spread is increasing in the dispersion of trader valuations.** At given prices, an increase in the inter-quartile range of  $F$  causes fewer limit orders to be submitted. The spread must widen to attract limit orders in order to clear the market for liquidity.

**The bid-ask spread encompasses the median trader.** For the quantity of the asset demanded to be equal to the quantity supplied, prices must encompass the median trader. This, together with the previous result, implies that as  $N \rightarrow \infty$ , both  $A$  and  $B$  tend to the valuation of the median trader.

## 5 Concavity of limit order utility

It was noted in the introduction that the analysis of trading under uncertainty would exploit the concavity of the function describing the payoff to limit orders. Like the utility function of a risk-averse agent, a concavity implies aversion to uncertainty. These themes are treated in a closed model in the next section. The purpose of this section is to prepare the ground by defining the shape of the relevant function.

The following lemma provides a mathematical characterization of the payoff to limit orders in the basic model. It relies on neither of the symmetry assumptions of Section 4.

**Lemma 5.1** *The expected utility of placing a limit bid to a trader with valuation  $V$  can be expressed as*

$$(V - B)Pr [T^s < D_\rho] = (V - B) \left( \frac{\mu_s^*}{\mu_s^* + \rho} \right)^{L_B + 1}. \quad (16)$$

This is a concave function of  $\mu_s^*$  when the expected time for the order to reach the top of the bid queue,  $L_B/\mu_s^*$ , is less than twice the expected time to the deadline,  $1/\rho$ , i.e. when

$$\mu_s^* > L_B\rho/2. \quad (17)$$

An analogous result holds on the sell side.

**Proof.**  $T^s$  is the time to the  $(L_B + 1)$ th sell market order. Sell market orders follow a Poisson process of intensity  $\mu_s^*$ . Recall if two Poisson processes have intensities  $\beta$  and  $\gamma$ , the probability at time zero that the first event to occur belongs to the process with intensity  $\beta$  is

$$\frac{\beta}{\beta + \gamma}.$$

Write  $T_i^s$  for the  $i$ th instance of the market sell order process. Then

$$\begin{aligned} \Pr[T^s < D_\rho] &= \Pr[T_{(L_B+1)}^s < D_\rho] \\ &= \Pr[T_{L_B}^s < D_\rho] \Pr[T_{(L_B+1)}^s < D_\rho | T_{L_B}^s < D_\rho] \\ &= \Pr[T_{L_B}^s < D_\rho] \left( \frac{\mu_s^*}{\mu_s^* + \rho} \right) \\ &= \left( \frac{\mu_s^*}{\mu_s^* + \rho} \right)^{L_B+1}, \end{aligned}$$

where the third line follows by applying the standard result reviewed above to the independent durations of the market sale arrival process and the fourth line follows by induction. The concavity property follows by differentiating this function twice. Figure 2 provides an illustration. ■

## 6 Market uncertainty

Sections 2 and 3 develop a basic model in which the number and distribution of trader types are common knowledge to traders. This section relaxes the common knowledge assumption. It introduces market uncertainty into the model – that is, uncertainty on the part of traders about “who else is in the market”. However, it assumes that traders cannot usefully update their priors about the market by observing the order flow. It solves for equilibrium in a symmetric case and compares this to the benchmark case of Section 4. For reasonable parameter values, spreads in fact widen under this uncertainty.

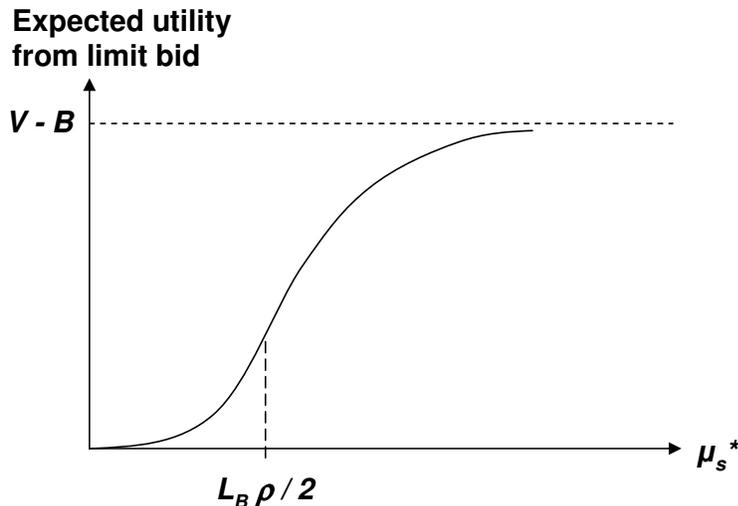


Figure 2: When  $\mu_s^* > L_B \rho / 2$  the payoff to limit orders is concave in anticipated market order intensity.

Traders report a preference for “taking what liquidity is there” rather than offering liquidity, at times when the order flow in the near future is uncertain. This may be a feature of morning trading or trading following a break such as a holiday. It is also characteristic of the period in London before the New York Stock Exchange opens, or prior to low content news events such as the results of sports fixtures. This section helps explain wide spreads and aversion to supplying liquidity whenever traders face uncertainty of this sort. Section 7 will elaborate the model still further replacing this uncertainty with uncertainty of a quite different type: uncertainty that can be learned about and resolved by traders as they participate in the market. In that setting, the results of this section are reversed.

This section drops the assumption of Section 2 that  $\Theta$  is a singleton containing the truth,  $(N, F)$ , to which all traders attach probability one. Instead,  $\Theta$  covers a range of possibilities, and traders share a prior distribution of beliefs over its elements,  $\mathbf{P}$ . Whereas the expected utility of a market purchase,  $(V - A)$ , is insensitive to  $\mathbf{P}$ , the expected utility from a limit bid is not. Thus the expected utility of a limit bid is

$$(V - B) E_{\mathbf{P}} \{ \Pr [ T^s < D_\rho ] \}. \quad (18)$$

This expression is in general different to the benchmark payoff derived by taking traders’ unconditional priors over states of the market,  $(N, F)$ ,

$$(V - B) \Pr [ T^s < D_\rho | (N, F) ]. \quad (19)$$

The expressions (18) and (19) describe the payoff to limit orders in an uncertain and a certain case respectively. However, they share the same constant unconditional prior over trader types. Thus, they differ only in the extent to which traders are uncertain over  $\Theta$ . The following example investigates their relative magnitudes. It shows that under normal trading conditions (when queues are not extremely long or traders not extremely impatient), (18) has a lower magnitude than (19) due to the concavity illustrated in Figure 2. This is the sense in which limit order bidders are averse to uncertainty.

## 6.1 Uncertainty aversion: a symmetric example

This subsection incorporates uncertainty into the symmetric equilibrium considered in Section 4. It begins with a definition of equilibrium in the context of uncertainty.

**Definition 1** *For all  $\varphi \in \Theta$ , denote traders' expectations for the arrival rates of limit orders and market orders by the four variables  $\mu_{b,\varphi}^*$ ,  $\mu_{s,\varphi}^*$ ,  $\lambda_{b,\varphi}^*$  and  $\lambda_{s,\varphi}^*$ .<sup>14</sup> Denote the aggregate intensities of orders in possible world  $\varphi$  by  $\mu_{b,\varphi}$ ,  $\mu_{s,\varphi}$ ,  $\lambda_{b,\varphi}$  and  $\lambda_{s,\varphi}$ . In equilibrium, as defined in Section 2, traders rationally anticipate one another's actions at every eventuality in  $\Theta$ , i.e. for all  $\varphi \in \Theta$ ,*

$$\mu_{b,\varphi} = \mu_{b,\varphi}^*, \quad \mu_{s,\varphi} = \mu_{s,\varphi}^*, \quad \lambda_{b,\varphi} = \lambda_{b,\varphi}^*, \quad \lambda_{s,\varphi} = \lambda_{s,\varphi}^*. \quad (20)$$

*In addition, prices move so that at the truth (only),*

$$\mu_b = \lambda_s, \quad \mu_s = \lambda_b, \quad (21)$$

*where  $\mu_b$  ( $\mu_s$ ) is the true intensity of market purchases (sales), as well as being their unconditional prior.*

It is worth noting that traders do not infer from the prices  $A$  and  $B$  that the true state of the market is  $(N, F)$ . Having defined equilibrium, the following lemma follows easily.

**Lemma 6.1** *In equilibrium, the uncertainty over  $\Theta$  induces a mean-preserving spread in traders' unconditional expectations of market order intensities. That is,*

$$\begin{aligned} E_{\mathcal{P}}\{\mu_{s,\varphi}\} &= \mu_s, \\ E_{\mathcal{P}}\{\mu_{b,\varphi}\} &= \mu_b. \end{aligned}$$

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<sup>14</sup>The notation is intended to be exactly analogous to that defined in Section 3.

**Proof.** Recall that  $F_\varphi$  and  $N_\varphi$  are independent, and that  $(N, F) = (N^*, F^*)$ . As in the basic model, each trader's optimization problem is invariant over time, and she never opts to cancel limit orders. Then,

$$\begin{aligned} E_{\mathbf{P}}\{\mu_{s,\varphi}\} &= E_{\mathbf{P}}[N_\varphi IF_\varphi(V_s^*)] \\ &= IE_{\mathbf{P}}[N_\varphi]E_{\mathbf{P}}[F_\varphi(V_s^*)] \\ &= NIF(V_s^*) \\ &= \mu_s. \end{aligned}$$

Equivalent reasoning holds on the buy side. ■

The next definition imposes the symmetry needed for symmetric equilibria to exist.

**Definition 2** Define the family  $\{F_\varphi : \varphi \in \Theta\}$  to be symmetric about its median,  $m$ , if traders assign the same probability to any upward deviation from  $F$  as they do to an equivalent downward deviation. More precisely, the condition is that for any  $\varphi_0 \in \Theta$  there exists a  $\varphi_1 \in \Theta$  of equal probability density to  $\varphi_1$  such that for all  $v$ ,  $1 - F_{\varphi_0}(v) = F_{\varphi_1}(2m - v)$ .

**Proposition 6.2** Suppose that  $F$  and  $\{F_\varphi : \varphi \in \Theta\}$  are symmetric about  $m$ , and both queue lengths equal to  $L > 0$ . Then in any symmetric equilibrium,

$$\frac{A + B}{2} = m, \quad (22)$$

$$A - B = \left[ F^{-1}\left(\frac{3}{4}\right) - F^{-1}\left(\frac{1}{4}\right) \right] \left[ \frac{1 - E_{\mathbf{P}}\{\Pr[T^s < D_\rho]\}}{1 + E_{\mathbf{P}}\{\Pr[T^s < D_\rho]\}} \right]. \quad (23)$$

In addition,  $T^s$ , like  $T^b$ , is distributed as the  $(L + 1)$ th event of a Poisson process of intensity  $\frac{NI}{4}$ .

**Proof.** The case so defined is completely symmetric so that (except for the level  $m$ ) buying and selling are interchangeable in its formulation. So, in any symmetric equilibrium,

$$E_{\mathbf{P}}\{\Pr[T^s < D_\rho]\} = E_{\mathbf{P}}\{\Pr[T^b < D_\rho]\}. \quad (24)$$

The type space divides into four quartiles representing the four types of order,

$$F(V_b^*) = \frac{3}{4}; F(V_s^*) = \frac{1}{4}; \quad (25)$$

$$\mu_b = \mu_s = \frac{NI}{4}. \quad (26)$$

The cost of immediate execution to the marginal price-taking buyer,  $V_b^*$ , is equal to the cost of delayed execution. That is,

$$(V_b^* - B)E_{\mathbf{P}}\{\Pr[T^s < D_\rho]\} = A - B, \quad (27)$$

and an equivalent statement holds on the sell side. Substituting (24) and (25) into (27), doing the same on the sell side, and rearranging the two resulting simultaneous equations proves the proposition. The algebra is exactly as in the proof to Proposition 4.1. ■

**Proposition 6.3** *Suppose that  $F$  and  $\{F_\varphi : \varphi \in \Theta\}$  are symmetric about  $m$ , and both queue lengths equal to  $L > 0$ . Then equilibrium spreads are wider than under full certainty (i.e. when  $\Theta$  is a singleton containing the truth) provided that  $L < \frac{NI}{2\rho}$ .*

**Proof.** Comparing (23) with (12), the bid-ask spread is narrower when  $\Theta$  is a singleton iff

$$E_{\mathbf{P}}\{\Pr[T^s < D_\rho]\} < \Pr[T^s < D_\rho|(N, F)]. \quad (28)$$

By Lemma 6.1  $\varphi$  induces a mean-preserving spread in  $\mu_{s,\varphi}$  about the truth,  $\mu_s$ . By (28) the bid-ask spread is wider if limit order placers are averse to this mean-preserving spread, but is narrower if they are attracted by it. This depends on whether the function  $\Pr[T^s < D_\rho]$  is concave or convex at  $\mu_s = \frac{NI}{4}$ . But by Lemma 5.1 it is concave whenever  $\mu_s > L\rho/2$ , that is, whenever  $L < \frac{NI}{2\rho}$ . ■

Whether  $L < \frac{NI}{2\rho}$  depends on the institutional rules of the market and the fineness of the price lattice, but for a wide range of reasonable parameter values  $\frac{NI}{2\rho}$  is large compared to  $L$ .<sup>15</sup> Under these circumstances traders are averse to uncertainty and market-clearing spreads widen under uncertainty. This equilibrium has the same comparative statics properties as those detailed for the symmetric equilibrium with complete information in Section 4.

## 7 Fleeting orders

By making only small adaptations to the modelling apparatus of previous sections, this part of the paper analyzes *fleeting orders* on a limit order book. As was noted in the

<sup>15</sup>For example, let  $N = 50$ ;  $\rho = 1$  per day;  $I = 10$  per day. Then the concavity condition is  $L < 250$ , which is consistent with the typical state of a limit order book.

introduction, the term “fleeting order” was first used in Hasbrouck and Saar (2002) to describe limit orders that are cancelled within two seconds. In their sample, as many as 27.7% of all limit orders were fleeting.

In the basic model, limit orders may be cancelled in favor of market orders, but since the trader’s optimization problem is invariant over time, traders never choose to cancel. This section proposes an adaptation to the basic model which makes cancellation attractive under certain conditions. It explains the empirical finding of Hasbrouck and Saar (2002) with the following account. In the sense of Section 6, traders arrive at the market uncertain of its state. However, traders quickly learn its true state simply by placing a limit order and watching the market evolve. Therefore traders, who, if fully informed, would have behaved as price-takers, prefer to place a limit order for a short time to check if queues are depleting very rapidly and their limit order would execute fast. They cancel their order and take the price on the other side of the book when they have ruled out states of the market containing many trading partners for them.

## 7.1 A model of fleeting orders

The symmetric example of Section 6.1 is adapted simply by assuming that although traders experience uncertainty over  $\Theta$  at the time of every order opportunity, the uncertainty persists only until the first market order arrives from the other side of the market, when traders find out that market conditions are in fact equal to their unconditional priors,  $(N, F)$ . The timing of the arrival of this information, to coincide with the first market order, is inessential, but simplifies the analysis greatly. The mechanics of the update performed by traders is also unimportant. The *essential* point is that traders quickly become informed about the market by participating in it. Indeed, they typically (in this model, necessarily) become informed more quickly than their limit orders are executed.

## 7.2 The marginal canceller of limit orders

Definition 1, which described the rational expectations of traders at equilibrium in Section 6, is also appropriate here. However, market clearing spreads are not as in Section 6, since the arrival of information may cause some traders to cancel, changing the relative intensities of limit orders and market orders. Since it is the only time that the parameters

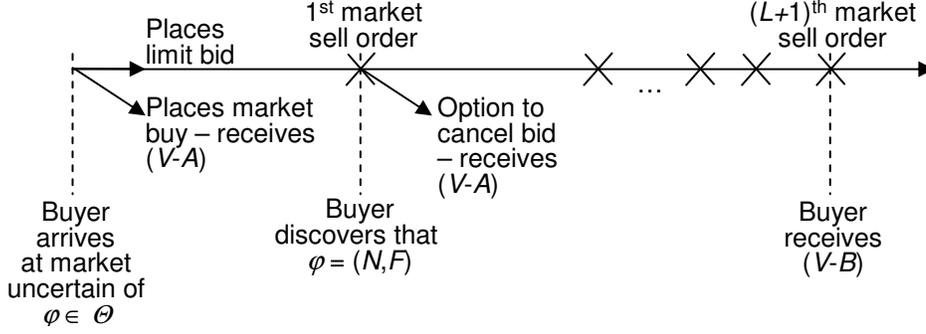


Figure 3: Illustration of the decision tree and information acquisition through time of a buyer (a trader with  $V > A$ ) in the model of fleeting orders. The crosses represent an instance of the stochastic arrival of market sell orders.

of a trader's optimization problem change, cancellation occurs at the arrival of the new information, or not at all. In addition, no limit order is cancelled simply to be replaced by a further limit order, for this would be to join the back of a queue in which the limit order has already advanced. Instead, cancellation, if it happens, is always in favor of a market order. These observations imply that the actions available to traders are as illustrated in Figure 3. They also motivate the following definition.

**Definition 3** *Define the marginal canceller of limit bids, of type  $V_b^*$ , to be the trader who, if she had initially placed a limit bid, is indifferent between cancelling and not cancelling, when she finds out that in fact market conditions are  $(N, F)$ . Define the marginal canceller of limit sales,  $V_s^*$ , analogously.<sup>16</sup>*

Any trader with  $V > V_b^*$  strictly prefers to cancel on finding out the truth, and may have preferred to have submitted a market purchase at the time of the order opportunity. In the certain world where  $\Theta$  is a singleton containing the truth, marginal cancellers of limit orders would not have placed a limit order in the first place since joining the back of an order queue is less attractive than persisting with it once one market order has arrived. To rule this out, a condition is imposed, so to speak bounding uncertainty from below.

**Condition 7.1** *In this section attention is restricted to markets such that types  $V_b^*$  and  $V_s^*$  submit limit orders at every order opportunity.*

Any trader with valuation higher than  $V_b^*$  eventually places a market buy, and cancels all limit bids. Similarly on the sell side. Therefore, applying again the aggregation principle

<sup>16</sup>For ease of notation, it is abused here, by using the same notation for the marginal cancellers as was used before for the marginal price takers.

of Section 3,<sup>17</sup>

$$\mu_s = NIF(V_s^*) \text{ and } \mu_b = NI[1 - F(V_b^*)]. \quad (29)$$

### 7.3 Symmetric equilibrium

**Proposition 7.2** *Suppose that in the model of Section 7.1 the symmetry assumptions of Proposition 6.2 as well as Condition 7.1 hold. Then in symmetric equilibrium the bid-ask spread is as wide as if there were no uncertainty [ $\Theta$  is a singleton containing the truth], but queue lengths were one shorter [ $L_A = L_B = (L - 1)$ ]. This is narrower than if there were no uncertainty.*

**Proof.** Marginal cancellers of limit bids are characterized by their indifference to the dilemma whether to cancel a limit bid when the first market sale arrives. In terms of payoffs, this is equivalent to the dilemma that a buyer would face who entered the market fully informed, but could enter the queue of waiting orders one ahead of the last place. An analogous argument holds on the sell side. Applying the symmetry of the equilibrium to (29),

$$F(V_b^*) = \frac{3}{4}; F(V_s^*) = \frac{1}{4}; \quad (30)$$

This equilibrium is therefore solved by the same series of equations as Proposition 4.1 with queue lengths set to  $(L - 1)$ . But in that setting, spreads increased with queue length, so spreads are narrower than if queue lengths had been  $L$ . ■

The comparative statics results of Section 4 continue to hold. Surprisingly, spreads are narrowed in the presence of rapidly resolved uncertainty, since traders ultimately submit orders as if the book were less deep than indeed it is. The next corollary shows that spreads narrow because this uncertainty draws out additional liquidity supply.

**Corollary 7.3** *Under the conditions of Proposition 7.2 some traders, who would not have submitted a limit order had they known  $(N, F)$  at the time of the order opportunity, are “drawn” into submitting limit orders which they do not cancel.*

**Proof.** The marginal canceller of limit bids would have placed a market order if she had known the true market conditions at the outset. By a continuity argument, there is a non-empty interval  $(V_b^* - \varepsilon, V_b^*)$  containing traders who would have done the same. Since

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<sup>17</sup>Second-order effects arising from dependence in the precise timing of cancellations where they are precipitated by the same market order are disregarded in this analysis.

their reservation value is less than  $V_b^*$  they do not cancel when they learn the true state of the market. A similar argument holds for  $V_s^*$ . ■

## 7.4 “Uncertainty loving” liquidity supply

This part shows that the concavity underlying the “uncertainty aversion” noted in Section 6 is here replaced by a convexity for some traders.

**Lemma 7.4** *The expected payoff to a trader with value  $V$  at the time of her order opportunity of submitting a limit bid is*

$$\Pr[T_1^s < D_\rho] E_{\mathcal{P}}\{\max\{(V - A), (V - B) \Pr[T_{L+1}^s < D_\rho | T_1^s < D_\rho]\}, \quad (31)$$

which is equal to

$$E_{\mathcal{P}}\{\max\{(V - A) \Pr[T_1^s < D_\rho], (V - B) \Pr[T_{L+1}^s < D_\rho]\}. \quad (32)$$

where  $T_i^s$  denotes the random time to the  $i$ 'th market sale. An analogous formula holds on the sell side.

**Proof.** Suppose that a trader with value  $V$  places a limit bid, and that the next market sale arrives before the trading deadline. The trader may then (by deciding whether to cancel) choose the maximum of  $(V - A)$  and  $(V - B) \Pr[T_{L+1}^s < D_\rho]$  in the full knowledge of  $\varphi$ . This is anticipated at the time of the order opportunity. The same holds for limit asks. ■

Figure 4 plots the value to the marginal canceller of limit bids,  $V_b^*$ , at the time of the order opportunity of placing a limit order conditional on  $\mu_{s,\varphi}$  (this is the contents of the expectation in (32)). It is the maximum of two schedules, one describing the case where the trader always cancels, the other showing the case where the trader never cancels. Where these two schedules coincide the payoff has a kink. For the marginal canceller, this occurs at exactly the truth,  $[\mu_{s,\varphi} = \frac{NI}{4}]$ .

The kink at  $\frac{NI}{4}$  reflects the optionality of placing a limit order. It implies a convexity in the expected payoff to limit orders, which was concave in the full information setting. Thus contrary to the “uncertainty aversion” exhibited by traders in Section 6 (with persistent uncertainty), trader  $V_b^*$  faces a convex payoff to limit orders, and is “uncertainty loving”.

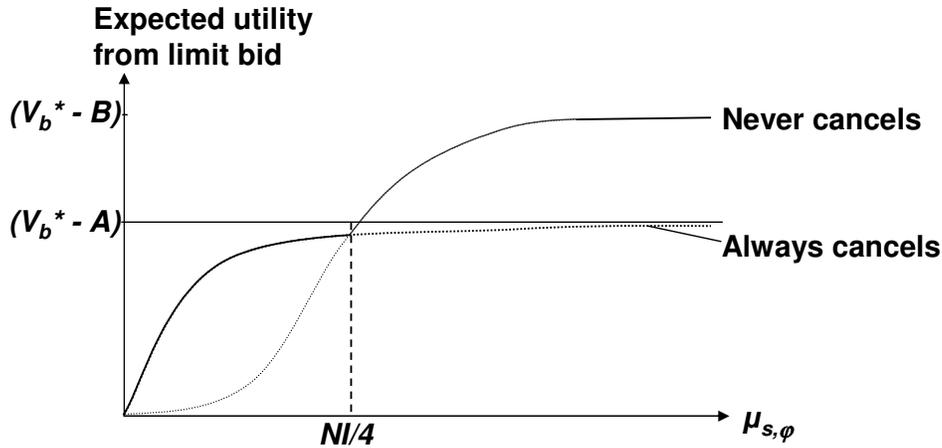


Figure 4: Shows the payoff to the marginal canceller of limit bids,  $V_b^*$ , of placing a limit order, conditional on  $\mu_{s,\varphi}$ . This is the maximum of two schedules (both dotted), one describing the case where the trader always cancels, the other showing the case where the trader never cancels.

## 7.5 The existence of fleeting orders

**Proposition 7.5** *Under the conditions of Proposition 7.2 some traders place fleeting orders. These orders are cancelled after an expected time of  $\frac{4}{NI}$ .*

**Proof.** Due to Condition 7.1, the value of (32) to the marginal canceller of limit bids exceeds  $(V_b^* - A)$ . Thus she reasons that she should join the queue in case there are many price-taking sellers, knowing she can cancel if she finds out there are not. By a continuity argument, there is a non-empty interval  $(V_b^*, V_b^* + \varepsilon)$  such that traders with reservation value in  $(V_b^*, V_b^* + \varepsilon)$  strictly prefer to submit limit orders. Since their values are greater than  $V_b^*$  they will also prefer to cancel in favor of a market order when they find out  $(N, F)$ . The expected time to cancellation is equal to the expected time to the first market order, which is  $\frac{4}{NI}$ . Analogous comments hold for  $V_s^*$ . ■

## 7.6 Overview of cancellation

The option to cancel together with uncertainty encourages some traders to place orders but cancel them fast as they find out more about the market. Others are “coaxed” into supplying liquidity where otherwise they would not. This generates extra liquidity supply which, unless spreads narrow, would exceed liquidity demand. The bid-ask spread must narrow to clear the market for liquidity. However, apart from this, the existence of fleeting orders does not change the results that the bid-ask spread widens with trader impatience,

narrows with trader numbers, widens with order book depth, and encompasses the median trader.

## 8 Discussion of the model

This section reviews some key assumptions of the models developed in this paper. It finishes with an assessment of the framework’s use in future research.

**The timing of traders’ actions is stochastic.** Traders on financial markets receive instructions from their clients or colleagues such as brokers whether to buy or sell a security and with what urgency. They manage a number of such tasks during the day. Their work-flow is constantly interrupted by unexpected events. The design of the model makes no assumption that traders can decide exactly when to trade. Instead, the parameter  $I$  is used as a measure of the trader’s activity on the market.<sup>18</sup>

**Traders have no budget or short-selling constraint.** Therefore, they do not face intertemporal trade-offs leading them to reduce intensities today to maximize the chance of making their trades at a time when prices will have moved to their advantage. It is better for traders to trade now, and then also to trade later if prices move to their advantage.

**Liquidity providers set prices competitively.** On any electronic limit order book, only a small number of points on the lattice of admissible prices are attractive to traders. Therefore queues of orders often develop on the most attractive prices. The model is appropriate for market conditions where the limit order flow is sufficiently large that there are queues of waiting orders at and around the bid and the ask. An assumption of the model is that how to participate in these order queues is a second order problem for traders. Thus the limit bidder’s choice whether to join the back of the queue of bids at

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<sup>18</sup>This means that orders arrive randomly in the model. This stochastic property has benefits in asymmetric and incomplete information settings. It implies that equilibria with asymmetric or incomplete information are not typically fully revealing. Thus, the model is flexible enough for market participants to make Bayesian updates conditional on history or on future trigger events. This is reminiscent of two strands in the literature introduced by Glosten and Milgrom (1985) and Kyle (1985) which introduce the necessary randomness for Bayesian update in a different way: by incorporating non-strategic “noise” traders who act randomly. See also Easley and O’Hara (1987), Easley and O’Hara (1992) and Admati and Pfleiderer (1988).

a more attractive price to her, or to jump the queue by placing a less attractive price, is marginal in comparison to the “make or take” decision. From this perspective, traders do not decide on the bid and ask prices but are faced by them exogenously. They make only two strategic choices: “buy” or “sell”; and “limit order” or “market order”.

The model therefore describes markets where liquidity providers face prices competitively rather than setting them strategically. Thus, bidders cannot shade their bids to exploit sellers who urgently need to trade. Doing so would condemn them to a disadvantageous price point at the back of the queue of outstanding bids. Instead, bidders respond to urgent selling competitively: seeing that urgent sellers are rapidly depleting the bid queue they are attracted to it. This is in contrast to the dynamic model of Foucault (1999) in which traders first decide whether to place a market order or a limit order; then, if a limit order is selected, at what price to make the limit order. In equilibrium, limit order traders select prices to be as unattractive as possible, subject to their acceptance by subsequent traders in certain desirable states of the world. Foucault’s equilibrium condition is unavailable here. Instead, this model uses an equilibrium concept where prices move in the long run to equalize the arrival rates of limit orders with market orders from the other side of the market.

**Order book depth is exogenous.** In the model, the determinants of order book depth are not studied. It remains exogenous and constant in equilibrium. Determining depth is a complex problem, which must take into account the fineness of the price lattice and other institutional features of the exchange. For example, once the bid queue becomes extremely long, the marginal bid is typically placed on a higher price tick rather than at the back of the queue. The threshold length at which a given bidder will choose a higher price tick is decreasing with the fineness of the price lattice.

This approach shifts the attention of the model away from the short run evolution of queue lengths to the definition of equilibrium spreads and average order arrival rates. Parlour (1998) examines the two-way causality between order flow and queue lengths. In particular, she shows how a market order, by depleting a queue of limit orders, makes it more likely that a new limit order will soon arrive at that queue. Equally, a limit order, by extending the queue it joins, makes future limit orders at that queue less frequent. Parlour (1998) also derives the less immediate result that changes in the length of the bid queue alter the arrival rates of orders at the ask and vice versa.

One interpretation of this assumption is that a well-stocked, benign market maker trades with every market order immediately and commits to trade with every limit bid (ask) when  $L_B + 1$  ( $L_A + 1$ ) future market sales (purchases) have arrived.

## 8.1 The model in future research

The model offers a number of fruitful avenues for future theoretical research. The assumption that queue lengths are constant can be relaxed by only a minor adjustment to the analytical framework. This would result in a dynamic model of the short-run that, similar to Parlour (1998), would reflect the effect of future changes in queue length on the trader's decision. However, whatever the values of  $A$  and  $B$  (within broad limits), the ergodic distribution of order queue lengths, endogenous on this approach, would then *itself* adjust to market-clearing levels, without any need for adjustment in prices. It is therefore difficult to marry this with a realistic account of price movement. Such an account should capture the tendency for prices to move when queues get very long or very short. It is likely that it would have to incorporate explicitly a discrete lattice of admissible prices. If successful this strategy would jointly specify stationary distributions of prices and queue lengths, as well as time-to-execution for limit orders in equilibrium. This is the approach taken in Hollifield, Miller, Sandas, and Slive (2003), although they do not solve the trader's optimization problem, but use it as a basis for econometric estimation.

The model also provides a framework for characterizing markets where traders have different levels of impatience. For example, a model could be investigated sharing the feature of Foucault, Kadan, and Kandel (2003) that traders draw from a set of two impatience types,  $\rho_1$  and  $\rho_2$ . Equally, traders could draw impatience levels from  $\mathbb{R}^+$ .<sup>19</sup>

Limit order books and other financial markets experience very frequent but irregularly spaced events, for example orders, mid-quote changes or trades. Recognizing this, much recent econometric work in market microstructure has used continuous time modelling techniques such as those developed in Engle and Russell (1998) or Bowsher (2003). Bowsher (2003) treats this data as a point process (a generalization of the Poisson process in which intensities vary with time and can be conditioned on event history) and estimates its intensity. However, the theoretical models whose predictions these papers have

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<sup>19</sup>Impatience could also be modelled as a function of trader reservation value:  $\rho(V)$  – possibly rising near the extremes of the distribution  $F$ .

tested, for example Kyle (1985), Admati and Pfleiderer (1988) and Easley and O'Hara (1992) are set in discrete time. By modelling Poisson processes in continuous time, the current paper offers a theoretical approach delivering results which are well adapted for testing with these techniques. Future research should focus on drawing out empirical predictions of the model, and understanding better in what respects Engle and Russell (1998) or Bowsher (2003) can be thought of as reduced form versions of it.

## 9 Conclusion

This paper presents a model where traders choose strategically between limit orders and market orders, but price limit orders competitively. They cannot decide exactly when to submit orders, but instead control Poisson order processes. The depth of the order book is exogenous to the model, but prices move to equate supply and demand for liquidity. The model brings into prominence the concavity of the expected utility of limit orders with respect to market conditions. Traders are uncertain of the number of other traders in the market, as well as of the distribution in their attributes. This makes them less inclined to provide liquidity, causing spreads to widen. However at times, especially in the very short term, traders face uncertainty which is resolved simply by watching and waiting. This causes some to submit *fleeting orders*. Liquidity supply increases: in fact, in equilibrium traders submit limit orders as if the order book were less deep.

Wide spreads have often been used as evidence of information effects. This paper indicates that even in the absence of future or current traders with an information advantage, pure uncertainty aversion is sufficient to widen spreads while the option to cancel can narrow them.

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