Title: The existence of equilibrium when excess demand obeys the weak axiom

Abstract: This paper gives a non-fixed point theoretic proof of equilibrium existence when the excess demand function of an exchange economy obeys the weak axiom. The proof is simple and geometrically intuitive, and it also permits a weakening of the continuity assumption on the excess demand function. (This paper is a more developed and (one hopes) improved version of an earlier working paper entitled *An Elementary Equilibrium Existence Theorem*, Working Paper W6 (2000), Nuffield College, Oxford.)

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1. Introduction

It is well known that while there are different ways of establishing the existence of competitive equilibria, all the classic proofs employ Kakutani's fixed point theorem (see Debreu [2]). This includes the 'excess demand approach' which solves this problem by constructing the economy's excess demand correspondence and then showing that there is a price at which excess demand can be zero. At the heart of this approach is a technical result known as the Gale-Nikaido-Debreu lemma, the proof of which uses Kakutani's fixed point theorem. If one is not willing to assume that excess demand has any other structural property besides Walras' Law, then the use of Kakutani's fixed point theorem is necessary, because in this case one could *prove* the fixed point theorem by assuming the Gale-Nikaido-Debreu lemma. (This observation is due to Uzawa, a proof can be found in Debreu [2].)

The view that an economy's excess demand function has no structure except Walras' Law is largely borne out by the indeterminacy theorems of Sonnenschein, Mantel and Debreu (see Mas-Colell et al [8]), but these theorems do rely on the absence of restrictions (or rather suitable restrictions) on preferences, endowments, and their distribution. When suitable restrictions are in place, the excess demand function *will* exhibit properties like gross substitutability and the weak axiom. For example, a model guaranteeing the former property can be found in Grandmont [3], while Quah [10] has a model with the latter property; the common theoretical features of these and other related models is ably examined in Jerison [5]. (See also Hildenbrand [4] for related empirical work.)

The significance of these aggregation results rests in part on the fact that the positive theory of general equilibrium is most satisfactory when the economy's excess demand function obey properties like the weak axiom. An economy with such a property will typically have a unique equilibrium price which is also stable with respect to the Walras' tatonnement [8]. They also exhibit nice comparative statics when subjected to endowment or other types of perturbations [9,11,13]. It turns out that the proofs of equilibrium existence under these types of structural assumptions are also more straightforward. It is known that there is a simple proof of equilibrium existence when the excess demand function obeys gross substitutability (see Barbolla and Corchon [2]). In this note, we give a proof of equilibrium existence using the separating hyperplane theorem, under the added assumption that excess demand satisfies the weak axiom.[1]

The equilibrium existence proof we give has the virtue that it separates very sharply the function of the geometric and continuity properties of excess demand. Provided an excess demand function satisfies the weak axiom, there will be some price vector with the following property: holding all other prices fixed, raising the price of good i leads to excess supply, and lowering it leads to excess demand. To guarantee the existence of such a price vector, which we call a *switching price*, the continuity of excess demand is not needed. So a switching price exists even when, say, the presence of indivisible goods leads to discontinuities in the excess demand function. Continuity is only needed if we wish to guarantee that the switching price is also an equilibrium price, and even then the continuity assumption can be significantly weakened.

In addition to being interesting in its own right, our equilibrium existence result has at least one other use. In the theory of nontransitive preferences, the problem of guaranteeing the existence of a demand bundle at any price-income situation is nontrivial and is usually solved by appealing to fixed point arguments or something equivalent (see for example, Shafer [14]). One can show that provided the preference is complete, the correspondence which associates each bundle on the budget plane to its supporting prices has a weak axiom-type structure, and the problem of finding the demand on this budget plane is formally equivalent to that of finding the equilibrium price of an excess demand function. So in this way, one also obtains an elementary proof of the existence of demand when an agent's preference is complete but not necessarily transitive (see Quah [12] for the details).

The next section presents our findings, which are organized around three major results: the first establishes the existence of a switching price, the second is a special case of the Gale-Nikaido-Debreu lemma, and the third is an equilibrium existence theorem. It also discusses the work of Abraham Wald, whose equilibrium existence theorem predated the work of Arrow, Debreu and McKenzie, and who uses a weak axiom-type condition in his proof. We rely here on John's [6] modern explication of Wald's model.

2. The Existence of Equilibrium

Consider an economy with l commodities. A common approach to the equilibrium existence problem involves the construction of a correspondence, $Z : V \to R^l$, where V, a subset of R^l, is the set of price vectors at which the excess demand Z is well-defined. Standard primitive conditions on technology and preferences will guarantee that V is nonempty, 0 is not in V, and that V is a *convex and pointed cone*; it is a convex cone in the sense that whenever p and p' are in V, so is $\lambda p + \lambda' p'$ where λ and λ' are positive scalars, and it is pointed in the sense that whenever p is in V, $-p$ is not in V. We maintain these assumptions. In addition, Z will have a number of properties:

Property 1. Z satisfies Walras' Law, i.e., $p \cdot Z(p) = 0$ for all p in P.

Property 2. Z is convex valued and upper hemicontinuous correspondence.[2]

For an exchange economy, under standard assumptions on agents' preferences, $V = R^l_{++}$

and Z will have two other properties:

Property 3. Z is bounded below.

Property 4. Z satisfies the following boundary condition: if p_n in R^l_{++} tends to $\bar{p}$ on the boundary of R^l_{++}, $\bar{p} \neq 0$, then for any z_n in $Z(p_n)$, the sequence $|z_n|$ tends to infinity.

It is known that if Z has Properties 1 to 4, then an equilibrium price exists, i.e., there p^* at which $0 \in Z(p^*)$. This result is usually established with Kakutani's fixed point theorem.[3]

We will show that another intuitive and instructive method of proof is available if Z satisfies *the weak axiom* in addition to Properties 1 to 4. The usual definition of the weak axiom is applied to functions only and says that Z obeys the weak axiom if whenever $p \cdot Z(p') \leq 0$ and $Z(p) \neq Z(p')$, then $p' \cdot Z(p) > 0$ [8]. The definition we adopt throughout this paper is applicable to correspondences and is also weaker than the usual definition. We say that the correspondence Z obeys the weak axiom if, for some p and p' in V there is z' in $Z(p')$ such that $p \cdot z' \leq 0$, then $p' \cdot Z(p) \geq 0$. We begin with a lemma which guarantees that a *finite* set of excess demand vectors must have a supporting price.

Lemma 1. *Suppose that the correspondence $Z : V \to R^l$ satisfies Property 1 and the weak axiom. Then for any finite set $S = \{z_1, z_2, ..., z_n\}$ where z_i is an element of $Z(p_i)$, there is x^*, in the convex hull of $\{p_1, p_2, ...p_n\}$ such that $x^* \cdot S \geq 0$.*

Proof. We proof by induction on n. If $n = 1$, choose $x^* = p_1$. If $n = 2$, then either $p_2 \cdot z_1$ or $p_1 \cdot z_2$ is non-negative. If it is the latter, choose, $x^* = p_1$. Assume now that the proposition is true for n and assume that it is not true for $n + 1$. Consider the following constrained maximization problem: $\max x \cdot z_k$ *subject to x satisfying conditions (a) $x \cdot z_i \geq 0$ for i in $I_k = \{1, 2, ..., k-1, k+1, ..., n+1\}$ and (b) x is in C, the convex hull of* $\{p_1, p_2, ...p_n, p_{n+1}\}$. By varying k, we have $n + 1$ problems of this sort.

Consider the case when $k = n+1$. By the induction hypothesis, there is certainly x such that $x \cdot z_i \geq 0$ for all i in I_{n+1}, since this set has only n elements. Furthermore, C is compact, so the problem has at least one solution, which we denote by $\bar{x}_{n+1}$. Since we are proving by contradiction, we assume that $\bar{x}_{n+1} \cdot z_{n+1} < 0$.

We will now show that $\bar{x}_{n+1} \cdot z_i = 0$ for all i in I_{n+1}. If not, the set $J = \{i : \bar{x}_{n+1} \cdot z_i = 0\} \cup \{n+1\}$ has n elements or less, and so there is $\bar{y}$ in C with $\bar{y} \cdot z_i \geq 0$ for all i in J. Consider now the vector $\theta\bar{y} + (1-\theta)\bar{x}_{n+1}$, which is also in C, provided θ is in $[0,1]$. Then

(i) $[\theta\bar{y} + (1-\theta)\bar{x}_{n+1}] \cdot z_i \geq 0$, for i in $J \setminus \{n+1\}$

(ii) $[\theta\bar{y} + (1-\theta)\bar{x}_{n+1}] \cdot z_i > 0$, for $i \notin J$ provided θ is sufficiently small

(iii) $[\theta\bar{y} + (1-\theta)\bar{x}_{n+1}] \cdot z_{n+1} \geq (1-\theta)\bar{x}_{n+1} \cdot z_{n+1} > \bar{x}_{n+1} \cdot z_{n+1}$.

This means that $\bar{x}_{n+1}$ does not solve the constrained maximization problem.

So the solution to this problem, $\bar{x}_{n+1}$, must satisfy $\bar{x}_{n+1} \cdot z_i = 0$ for i in I_{n+1} and $\bar{x}_{n+1} \cdot z_{n+1} < 0$. We can apply the same argument to a solution of the other problems. In this way, we obtain $\bar{x}_k$, for $k = 1, 2, ..., n+1$ with $\bar{x}_k \cdot z_i = 0$ for i in I_k and $\bar{x}_k \cdot z_k < 0$. Define $\bar{x} = [\sum_{i=1}^{n+1} \bar{x}_i]/(n+1)$; $\bar{x}$ is certainly in C. Furthermore, $\bar{x} \cdot z_i < 0$, for $i = 1, 2, ..., n+1$. By the weak axiom, $p_i \cdot Z(\bar{x}) > 0$ for all i. Since $\bar{x}$ is in the convex hull of the p_is, we have $\bar{x} \cdot Z(\bar{x}) > 0$, which contradicts Walras' Law (Property 1). QED

This lemma leads intuitively to the next result.

Lemma 2. *Suppose that $Z : V \to R^l$ satisfies Property 1 and the weak axiom. Then there is $p^* \neq 0$ in the closure of V such that $(p^* - p) \cdot Z(p) \geq 0$ for all p in V.*

Proof: We claim that co$Z \cap V^* = \emptyset$, where co$Z$ is the convex hull of the set $\{Z(p) \in R^l : p \in V\}$ and $V^* = \{v \in R^l : v \cdot p < 0 \text{ for all } p \in V\}$. If not, we can find $\sum_{i=1}^{K} \beta_i z_i$ in V^*, where z_i is in $Z(p_i)$ for some p_i, and the β_is are non-negative numbers that add up to 1.

By Lemma 1, there is x in V, with $x \cdot z_i \geq 0$ for all i, and consequently, $x \cdot [\sum_{i=1}^{K} \beta_i z_i] \geq 0$, contradicting the definition of V^*. So our claim is true. The separating hyperplane theorem guarantees that there is $p^* \neq 0$ such that $p^* \cdot Z(p) \geq p^* \cdot V^*$. Since V^* is a cone, 0 is in the closure of V^*, so we obtain $p^* \cdot Z(p) \geq 0$ (equivalently by Walras' Law, $(p^* - p) \cdot Z(p) \geq 0$) for all p in V. Note also that $p^* \cdot V^*$ must be bounded above, which means, since V^* is a cone, that $p^* \cdot V^* \leq 0$ and consequently $p^* \cdot (\mathrm{cl}(V^*)) \leq 0$ where $\mathrm{cl}(V^*)$ denotes the closure of V^*. By Lemma A(iii) in the Appendix, p^* is in $\mathrm{cl}V$ QED

We refer to any $p^* \neq 0$ satisfying the property in Lemma 2 as a *switching price of Z*. The motivation for this term is quite clear. For a price p in V, with $p^i = p^{*i}$ for all i, except $i = k$, the switching property tells us that $(p^* - p) \cdot Z(p) = (p^{*k} - p^k) Z^k(p) \geq 0$. In other words, if p^k is greater than p^{*k} there will be excess supply of k; if it is lower, there will be excess demand of k. Of course, this motivation presupposes that p^* is in the interior of V, something which we have yet to prove. The next result shows that when $V = R^l_{++}$, this is guaranteed by Properties 3 and 4.

Lemma 3. *Suppose that $V = R^l_{++}$ and let p^* in $R^l_+ \setminus \{0\}$ be a switching price of Z. If Z satisfies Properties 3 and 4 then p^* is in R^l_{++}.*

Proof. Since p^* is in R^l_+, p_n where $p^i_n = p^{*i} + 1/n$ is in R^l_{++} and so $Z(p_n)$ is non-empty. For some z_n in $Z(p_n)$, $(p^* - p_n) \cdot z_n = -[\sum_{i=1}^{l} z^i_n]/n \geq 0$ since p^* is a switching price. So $\sum_{i=1}^{l} z^i_n \leq 0$ for all n. This implies that p^* is in R^l_{++}. If not, since p_n tends to p^*, Property 4 says that $|z_n|$ tends to infinity while Property 3 says that z^i_n is uniformly bounded below, so $\sum_{i=1}^{l} z^i_n$ tends to infinity and will not be non-positive. QED

Combining Lemmas 2 and 3 gives us the next result. It is worth pointing out that this theorem on the existence of switching prices does not require the convex or compact valuedness of the excess demand correspondence, nor its upper hemicontinuity.

Theorem 1. *Suppose that the correspondence* $Z : R^l_{++} \to R^l$ *satisfies Property 1,3, and 4 and the weak axiom. Then there is* p^* *in* R^l_{++} *such that* $(p^* - p) \cdot Z(p) \geq 0$ *for all* p *in* R^l_{++}.

If we wish to go one step further and establish that a switching price is also an equilibrium price, then it should be quite clear that the upper hemicontinuity of Z will have to be brought into play.

Lemma 4. *Let* p^* *in* V *be a switching price of the correspondence* $Z : V \to R^l$. *If* Z *satisfies Property 2 then there exists* z^* *in* $Z(p^*)$ *such that* $z^* \cdot p \leq 0$ *for all* p *in* V. *In particular, if* p^* *is in the interior of* V *and Property 1 holds, then* $z^* = 0$.

Proof. Define $V^0 = \{v \in R^l : v \cdot p \leq 0 \text{ for all } p \in V\}$. The set V^0 is called the *polar cone* of V. By way of contradiction, suppose that $Z(p^*)$ and V^0 are disjoint. It is trivial to check that V^0 equals the polar cone of cl$V$, which we shall denote by $(\text{cl}V)^0$, so $Z(p^*)$ and $(\text{cl}V)^0$ must also be disjoint. By the separating hyperplane theorem, there is $v \neq 0$ such that $v \cdot Z(p^*) > v \cdot (\text{cl}V)^0$. Since 0 is in $(\text{cl}V)^0$, we have $v \cdot Z(p^*) > 0$. Furthermore, since $(\text{cl}V)^0$ is a cone and $v \cdot (\text{cl}V)^0$ is bounded above, $v \cdot (\text{cl}V)^0 \leq 0$. By Lemma A(ii) in the Appendix, v is in cl$V$. So there is a sequence v_n in V with v as its limit. Choose a sequence of positive scalars a_n tending to zero; then $p_n = a_n v_n + p^*$ tends to p^*, and since V is a convex cone p_n is in V. Since p^* is a switching price, we have $(p^* - p_n) \cdot Z(p_n) = -a_n v_n \cdot Z(p_n) \geq 0$ for all n. By Property 2, there is a sequence z_n, extracted from $Z(p_n)$, which converges to z^* in $Z(p^*)$. So $v \cdot z^* \leq 0$, which contradicts $v \cdot Z(p^*) > 0$. Thus we conclude that $Z(p^*)$ and V^0 are not disjoint.

Suppose p^* is in the interior of V and that $z^* \neq 0$. By Property 1, $z^* \cdot (p - p^*) \leq 0$ for p in V and in particular for p in a neighborhood of p^*. This is only possible if $z^* = 0$. QED

The next result follows from combining Lemmas 2 and 4.

Theorem 2. *Suppose that V is relatively closed in $R^l \setminus \{0\}$ and that $Z : V \to R^l$ satisfies Properties 1, 2, and the weak axiom. Then there is p^* in V and z^* in $Z(p^*)$ such that $z^* \cdot p \leq 0$ for all p in V.*

The claim in Theorem 2 is true even when the weak axiom is not assumed (while keeping the other assumptions). In that more general form, the result is known as the Gale-Nikaido-Debreu lemma and it is at the heart of existence proofs of general equilibrium using the excess demand approach. When the weak axiom is not assumed, its proof requires Kakutani's fixed point theorem and indeed it is known that the result *implies* Kakutani's fixed point theorem [2]; as we have seen, when the weak axiom is assumed an elementary proof which dispenses with fixed point arguments is available.

Up to this point, we have made use of Property 2 to guarantee that a switching price is indeed an equilibrium price. In the case when Z is a *function*, this continuity property can be weakened by excluding only certain types of discontinuities in Z's behavior. We say that a function $\phi : R_+ \to R$ has a *drop* at s if $\phi(s) > \sup\{\phi(s') : s' > s\}$.

Property 2′. $V = R^l_{++}$, $Z(p)$ is a singleton for all p in R^l_{++}, and $Z^i(\cdot|\bar{p}) : R_+ \to R$, given by $Z^i(p^i|\bar{p}) = Z^i(\bar{p}^1, \bar{p}^2, ..., p^i, \bar{p}^{i+1}, ..., \bar{p}^l)$ admits no drops for all i and $\bar{p}$ in R^l_{++}.

Lemma 5. *Suppose that $V = R^l_{++}$, $Z : R^l_{++} \to R^l$ satisfies Properties 1 and 2′ and there exists a switching price p^* in R^l_{++}. Then $Z(p^*) = 0$.*

Proof. Since $p^* \gg 0$, if $Z(p^*) \neq 0$, Walras' Law says that there is k with $Z^k(p^*) > 0$. By Property 2′, there is $p^k > p^{*k}$ such that $Z^k(p^k|p^*) = Z^k(\hat{p}) > 0$, where $\hat{p} = (p^{*1}, ..., p^k, p^{*(k+1)}, ..., p^{*l})$. Then $(p^* - \hat{p}) \cdot Z(\hat{p}) = (p^{*k} - p^k)Z^k(\hat{p}) < 0$, which is a contradiction. QED

Gathering together our results gives us the following equilibrium existence theorem.

Theorem 3. *Suppose that* $Z : R^l_{++} \to R^l$ *satisfies Properties 1, 2 (or 2′), 3, 4, and the weak axiom. Then there is a price* p^* *in* R^l_{++} *such that* $0 \in Z(p^*)$.

Proof. Lemma 2 guarantees that there is a switching price in R^l_+. Lemma 3 guarantees that p^* is in R^l_{++}. If Property 2 holds, Lemma 4 guarantees that $0 \in Z(p^*)$. If Property $2'$ holds, Lemma 5 guarantees that $Z(p^*) = 0$. QED

We conclude with some comments on the related literature. The first rigorous solution to the existence problem in a general equilibrium model is due Abraham Wald, who showed that an equilibrium exists in a Walras-Cassel production economy. In Wald's proof, the household sector of the economy has an aggregate demand function which is assumed to be invertible, with the inverse assumed to satisfy the weak axiom (see John [6]). Wald needs this assumption because his approach to the equilibrium existence problem requires that he solve what, in modern mathematical terms, is known as the Stampacchia variational inequality problem; assuming the weak axiom (as he does) allows for an elementary solution to this problem [6].

Our starting point is the excess demand approach to equilibrium existence as developed by Debreu and others. To the standard assumptions on the excess demand correspondence we add the weak axiom. At the heart of our approach is the search for a switching price; in the mathematical literature, this is a special case of the Minty variational inequality problem.[4] A discussion of this problem can be found in John [7] who also gives a proof of Lemma 1 using an elementary intersection property of convex sets. The solution we give here can also be modified in an obvious way to deal with the general Minty variational inequality problem.

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Appendix

For the sake of completeness, we include here a short proof of a simple result on cones we used in the proofs of Proposition 1 and Lemma 3. Suppose that A is a convex and pointed cone in R^l and write $A^* = \{v \in R^l : v \cdot a < 0 \text{ for all } a \in A, a \neq 0\}$, and $A^0 = \{v \in R^l : v \cdot a \leq 0 \text{ for all } a \in A, \}$. The set A^0 is usually referred to as the polar cone of A. A^* is defined similarly, except that the inequality is strict rather than weak. We denote the closure of any set S by cl$S$.

Lemma A. *(i) $cl(A^*) = (clA)^0$, (ii) If A is closed, $(A^0)^0 = A$, (iii) $(cl(A^*))^0 = clA$.*

Proof. (i) If x is in cl(A^*), there is x_n tending to x such that $x_n \cdot a < 0$ for all a in $A \setminus \{0\}$. Taking limits, we have $x \cdot \bar{a} \leq 0$ for all $\bar{a}$ in cl$A$. So x is in $(\text{cl}A)^0$. For the other inclusion, since $(\text{cl}A)^0 \subseteq A^0$, it suffices to show that $A^0 \subseteq \text{cl}(A^*)$. If x is in A^0, by definition, $x \cdot a \leq 0$ for all a in A. Since A is convex and pointed, by the separating hyperplane theorem, there is $w \neq 0$ such that $w \cdot A > 0$, for all a in $A \setminus \{0\}$. Since $[x - (w/n)] \cdot a < 0$ for all a, $x - (w/n)$ is in A^*. Letting n go to infinity, we see that x is in cl(A^*).

(ii) If a is in A, for all v in A^0, $v \cdot a \leq 0$, so a is certainly in $(A^0)^0$. On the other hand, if a is not in A, then by the separating hyperplane theorem, there is w such that $w \cdot a > w \cdot A$. (Note that the inequality is strict because A is closed.) This means that $w \cdot A \leq 0$; otherwise the right hand side of the inequality is unbounded above. So w is A^0. We also have $w \cdot a > 0$, so this means that a is not in $(A^0)^0$.

(iii) This clearly follows from (i) and (ii). QED

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Footnotes:

1. Something roughly analogous is true of the existence proofs in game theory. The standard proof of the existence of Nash equilibria uses Kakutani's fixed point theorem, but for special cases like zero-sum games or supermodular games one can appeal, respectively, to the separating hyperplane theorem and Tarski's fixed point theorem.

2. A correspondence $F : X \to Y$ where X and Y are metric spaces is *upper hemicontinuous* if, for every sequence x_n tending to x in X, and every sequence y_n in $F(x_n)$, there is a subsequence of y_n with a limit in $F(x)$. In particular, this implies that $F(x)$ is compact for every x in X.

3. All the claims made in this paragraph can be verified in Debreu [2].

4. I am very grateful to Reinhard John for pointing this out to me.