TROPICAL GEOMETRY TO ANALYSE DEMAND

Elizabeth Baldwin* and Paul Klemperer†

preliminary draft May 2012, slightly revised July 2012

The latest version of this paper and related material will be at
http://users.ox.ac.uk/∼wadh1180/ and www.paulklemperer.org

ABSTRACT

Duality techniques from convex geometry, extended by the recently-developed mathematics of tropical geometry, provide a powerful lens to study demand. Any consumer's preferences can be represented as a *tropical hypersurface* in price space. Examining the hypersurface quickly reveals whether preferences represent substitutes, complements, "strong substitutes", etc. We propose a new framework for understanding demand which both incorporates existing definitions, and permits additional distinctions. The theory of *tropical intersection multiplicities* yields necessary and sufficient conditions for the existence of a competitive equilibrium for indivisible goods–our theorem both encompasses and extends existing results. Similar analysis underpins Klemperer's (2008) *Product-Mix Auction*, introduced by the Bank of England in the financial crisis.

JEL nos:

Keywords:

Acknowledgements to be completed later

*Balliol College, Oxford University, England. elizabeth.baldwin@economics.ox.ac.uk
†Nuffield College, Oxford University, England. paul.klemperer@economics.ox.ac.uk

1 Introduction

This paper introduces a new way to think about economic agents' individual and aggregate demands for indivisible goods,[1] and provides a new set of geometric tools to use for this.

Economists mostly think about agents' demands by focusing on the direct utility functions. We instead begin by focusing on the geometric structure of the regions of price space in which an agent demands different bundles. Our crucial observation is that dividing price space in this way creates *precisely* the geometric structure which is studied in the recently-developed, non-Euclidean, branch of algebraic geometry called "tropical geometry".[2] We can therefore use the tools of convex and tropical geometry, such as the duality between the geometric structure of an agent's demand in price space and the same agent's demand in quantity space, to obtain new insights about demand. Moving backwards and forwards between the dual representations of demand in price space and quantity space improves our understanding of both.

For example, it is much easier to aggregate individual demands in price space, but translating aggregate demand back into quantity space allows us to prove a strong theorem that encompasses and extends existing results about when a competitive equilibrium exists.

On the other hand, if we start from the (direct) valuation function in quantity space, our methods for translating to the dual in price space quickly reveal the key properties of demand. Many existing results in demand theory can be understood more readily, and developed more efficiently, using our tropical-geometric perspective than using existing methods.

Examining the geometric structure of demands also suggests a natural way of classifying them: we say two demand "types" are the same if certain sets of vectors associated with their "tropical hypersurfaces" are the same. It is then elementary to check, using simple rules about the signs and magnitudes of the entries in these sets of vectors, whether a demand type is, for example, substitutes, or complements, or "strong substitutes" (Milgrom and Strulovici, 2009), or "gross substitutes and complements" (Sun and Yang, 2006), etc., etc. So our approach provides an easy test of the nature of preferences. Examining the vectors also reveals new results about categories of demands, and clarifies relationships among them. For example, it makes clear why the conditions for all of three or more indivisible goods to be (ordinary) substitutes are far more restrictive than the conditions for all of them to be complements–although these conditions are of course symmetric in the two-good case.

Understanding the *aggregate* demand of multiple agents entails intersecting their "tropical hypersurfaces", so the theory of "intersection multiplicities" yields a simple necessary and sufficient condition on preferences that guarantees the existence of a competitive equilibrium for indivisible goods. The condition can easily be checked from the determinants of the vectors describing the demand, and shows new demand structures

[1] Baldwin and Klemperer (in preparation) show the relevance of our techniques to analysing *divisible* goods also, in contexts such as the Product-Mix Auction.

[2] This assumes, as is standard in the indivisible-goods literature, that preferences are quasilinear. Tropical geometry was developed by, among others, Mikhalkin (2004, 2005). We believe it has not previously been applied to economics. Goeree and Kushnir (2011) have recently used techniques of convex analysis (see, e.g., Rockafellar, 1970), upon which tropical geometry builds, in a different context.

which guarantee equilibrium existence. For example, we exhibit a previously-unanalysed demand type involving only complementary relationships between goods, in which equilibrium always exists.[3]

Furthermore, it is straightforward in our framework that properties such as the existence of equilibrium are preserved under (unimodular) basis changes of these same vectors. Using this observation reveals when important properties of demands are the same.

Finally, our understanding of the convex- and tropical-geometric structure of agents' preferences facilitates the analysis of "Product-Mix Auctions" (Klemperer, 2008, 2010; Baldwin and Klemperer, in preparation).[4] In these auctions–introduced by the Bank of England in response to the 2007 Northern Rock bank run and the subsequent financial crisis–bidders offer prices for alternative bundles of goods, so their bids can be represented geometrically as sets of points in multi-dimensional price space.[5] Our geometric techniques tell us what kinds of bids are needed to represent different kinds of preferences, what "coherent" bids look like, how to efficiently solve for equilibrium (and when it exists), etc.[6]

We begin, therefore, by explaining the basic concepts of tropical geometry. Section 2 describes the properties of a "tropical hypersurface", a geometric object which contains precisely those points at which the agent is indifferent between two or more bundles and which therefore contains enough information to fully describe an agent's valuation function. It is composed of linear pieces known as 'facets' which separate the price space into regions in which the agent demands different bundles.

Section 3 explores duality in our context. The same set of vectors that are orthogonal to the facets of the tropical hypersurface also generates the surface of the agent's valuation function in quantity space (strictly, it generates the convex hull of that surface). So there is a precise correspondence between classes of tropical hypersurfaces (in price space) and subdivisions of "Newton polytopes" (in quantity space).

Section 4 focuses further on the structure of individual demand, by defining a "type" of demand by the same set of vectors as above. Since these vectors describe the ways in which the bundles demanded by the agent change with prices, they identify the key

[3]The demand type is fundamentally different from (i.e., not simply a basis change of) strong substitutes, unlike, e.g., "gross substitutes and complements"—see below.

[4]Product-mix auctions are "one-shot" auctions for allocating heterogeneous goods. Their equilibrium allocations and prices are similar to those of Simultaneous Multiple-Round Auctions in private-value contexts, but they permit the bid-taker to express richer preferences, are more robust against collusive and/or predatory behaviour, and are, of course, much faster.

[5]Bids are made as lists of coordinates in implementations like the Bank of England's; the Bank itself (the bid-taker) depicts these bids, and also its own preferences, geometrically.

The Bank has already successfully auctioned approaching £100 billion in funds, and an Executive Director of the Bank of England (Paul Fisher) described the auction design as "A world first in central banking...potentially a major step forward in practical policies to support financial stability" (Milnes, 2010). (In principle, of course, funds are almost continuously divisible, but we can apply our same indivisible-good duality techniques.)

[6]Expressing even richer preferences, and over more goods, than the Bank of England's current implementation permits may in some circumstances be important to this or other Central Banks who have shown interest in using the auction, or for other applications such as the sale of related products by a manufacturer, the purchase of electricity generated in different locations, the trading of permits for emission reductions relating to different kinds of deforestation, etc. Our geometric methods also permit easy alternative ways of representing preferences as bids.

characteristics of demand. Our representation permits the easy proof and interpretation of existing results in the theory of demand.

Section 5 turns to the structure of aggregate demand. Tropical intersection theory inspires our proof that with n indivisible goods, a competitive equilibrium always exists for concave demands of a type, if and only if the determinant of every subset of n of the vectors that define the type has determinant 0, +1, or −1 (plus an additional condition for cases in which the set of aggregate demands is in fewer dimensions than the number of goods). An easy corollary is that with three or fewer goods, such equilibrium always exists if and only if goods are "strong substitutes" in Milgrom and Strulovici's (2009) terminology, or a basis change of strong substitutes. But in four or more dimensions the condition for equilibrium existence is strictly weaker; we provide an example. Our theorem thus significantly extends earlier results; indeed a number of existing results follow immediately from ours.[7]

Finally, we observe that since it is straightforward to "add" tropical hypersurfaces in price space, a natural and easy way to compute aggregate demand from agents' direct utility functions is by first computing each agent's tropical hypersurface. This is essentially a generalisation of the point that it is easy to compute total demand from individuals' bids in a Product-Mix Auction. However, we defer substantive discussion of the application of tropical geometry to Product-Mix Auctions to Baldwin and Klemperer (in preparation). So we conclude in Section 6.

2 Representing Demand in Tropical Geometry

2.1 Assumptions and Motivation

There are n goods, which come in indivisible units. Each agent has a valuation function $u : A \to \mathbb{R}$ over a finite set $A \subsetneq \mathbb{Z}^n$ of possible bundles. We permit negative bundles to allow consideration of sellers as well as buyers. Agents have quasilinear preferences (and so, for example, no budget constraints). The price vector is $\mathbf{p} \in \mathbb{R}^n$, so different units of the same good always have the same price.[8] So the agent's demand set is

$$D_u(\mathbf{p}) := \arg\max_{\mathbf{x} \in A} \{u(\mathbf{x}) - \mathbf{p}.\mathbf{x}\}.$$

We are interested in how $D_u(\mathbf{p})$ varies with $\mathbf{p}$. It is of course constant while it is single-valued. All the action takes place at those $\mathbf{p}$ at which more than one bundle is

[7] Easy corollaries of our theorem include: Milgrom and Strulovici's (2009) result that equilibrium exists for all "strong substitutes" demands, and Sun and Yang's (2006) result about the existence of equilibrium in their "two-group gross substitutes and complements" economy (both of which are generalisations of Kelso and Crawford's (1982) results); Hatfield et. al (2012)'s result about when a stable outcome is not guaranteed in a trading network; and Teytelboym (2012)'s proof of equilibrium existence in his model of contracts and trading on networks; as well as extensions of many of these results.

Our theorem identifies the classes of valuation functions for which competitive equilibrium is guaranteed. The *necessity* of this condition contrasts with previous results that show only that equilibrium always exists if all agents' valuation functions have a certain property, but may fail if exactly one valuation function does not have this property.

[8] We can, of course, model different units of a homogeneous good which are priced independently, by simply treating them as different goods.

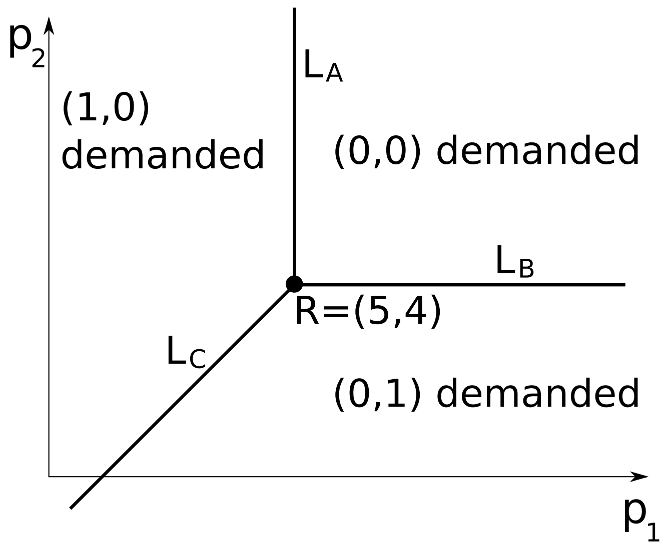


Figure 1: A simple tropical hypersurface (TH). The bundle demanded on each side of the TH is labelled.

demanded. So this set of prices is our principal object of study. We write this set of prices as[9]

$$\mathcal{T}_u := \{\mathbf{p} \in \mathbb{R}^n \mid \#D_u(\mathbf{p}) > 1\}\,. \tag{1}$$

The object $\mathcal{T}_u$ (with some additional structure–see Definition 2.3) is a convex-geometric object, known as a 'tropical hypersurface' (TH) in the new sub-discipline of algebraic geometry known as tropical geometry.[10]

A simple example is shown in Figure 1. The agent's valuations are $u(0,0) = 0$, $u(1,0) = 5$ and $u(0,1) = 4$. So its demand is for precisely one of these bundles in each of the three regions labelled, but switches from one bundle to another along the lines drawn.

The following subsections describe properties of THs, and also how the structure of the agents' demands can be recovered from them.

2.2 The Tropical Hypersurface: associating geometric objects with demand

We start by considering the local structure of a TH. Given a price $\mathbf{p}$ and its demand set $D_u(\mathbf{p})$, we ask for what other prices $\mathbf{p}'$ the demand set is the same, or closely related.

Definition 2.1.

1. The *cell interior* of the TH $\mathcal{T}_u$ at a price $\mathbf{p}$ consists of points $\mathbf{p}'$ such that $D_u(\mathbf{p}) = D_u(\mathbf{p}')$. A subset of $\mathcal{T}_u$ is a cell interior if it is the cell interior at some point in $\mathcal{T}_u$.

[9] We follow the mathematical literature in this slight abuse of notation.

[10] See Mikhalkin (2004) and others. In fact, Mikhalkin (2004) takes the tropical hypersurface associated to u to be the non-smooth locus of $\mathbf{p} \mapsto \max_{\mathbf{x}\in A}\{\mathbf{x}.\mathbf{p} - u(\mathbf{x})\}$. Thus our tropical hypersurfaces are 'upside down' compared with his. Mikhalkin's convention is not universal; Maclagan and Sturmfels (2009) take the non-smooth locus of $\mathbf{p} \mapsto \min_{\mathbf{x}\in A}\{u(\mathbf{x}) + \mathbf{x}.\mathbf{p}\}$, which defines tropical hypersurfaces the 'same way up' as ours, albeit shifted. Our convention seems the natural one from an economic point of view: we maximise surplus, being the value of a bundle minus its cost.

2. A subset of $\mathcal{T}_u$ is a *cell* if it the closure of a cell interior of $\mathcal{T}_u$.

3. The *affine span* of a cell of $\mathcal{T}_u$ is the smallest affine space containing the cell.[11]

4. The *boundary* of a cell of $\mathcal{T}_u$ consists of those points in the cell that are not in its cell interior.

Note that the cell interior is the largest set that is both contained in the cell and open in the affine span of the cell.[12]

We call a cell of dimension k a *k-cell*,[13] and call an $(n-1)$-cell a *facet*.

Figure 1 illustrates these concepts. The three line-segments L_A, L_B and L_C in the figure do *not* include the point R. Each of these line-segments is a cell interior: $D_u(\mathbf{p}) = \{(0,0),(1,0)\}$ in L_A, $D_u(\mathbf{p}) = \{(0,0),(0,1)\}$ in L_B, and $D_u(\mathbf{p}) = \{(1,0),(0,1)\}$ in L_C. The point R is also a cell interior: $D_u(R) = \{(0,0),(1,0),(0,1)\}$. The corresponding cells are the unions of these cell interiors with their limit points: $L_A \cup R$ is thus a cell, and indeed a facet; so are $L_B \cup R$ and $L_C \cup R$. Finally, R itself is a 0-cell.

The price R is also the boundary of each of the 1-cells $L_A \cup \{R\}, L_B \cup \{R\}, L_C \cup \{R\}$. (The 0-cell R has no boundary.) Note that the price R is contained in four cells, but each price in the TH is contained in precisely one cell interior. Finally, the affine span of any cell is the set of all prices at which the agent is indifferent between all the bundles in the cell, so the affine spans of $L_A \cup R$, $L_B \cup R$, and $L_C \cup R$, are the entire lines containing those line-segments, while the affine span of R is the point R itself.

It is immediate that:

I There are finitely many distinct cells, and the TH is the union of these.

II The cell interiors do not intersect.

Figure 2 illustrates the latter point: although the TH is 'two line segments crossing at a point', it has *four* 1-cells with distinct interiors (and also a single 0-cell at R).

Furthermore Definition 2.1 implies that for a price $\mathbf{p}'$ to be in the cell interior corresponding to a set of bundles $D_u(\mathbf{p})$, the agent must be indifferent between those bundles, that is, $\mathbf{p}'.(\mathbf{x}-\mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$, and the agent must strictly prefer these bundles to all others, that is, $\mathbf{p}'.(\mathbf{x}-\mathbf{x}'') < u(\mathbf{x}) - u(\mathbf{x}'')$ for all $\mathbf{x} \in D_u(\mathbf{p})$ and $\mathbf{x}'' \in A \backslash D_u(\mathbf{p})$. The cell corresponding to this cell interior contains its limit points, so a price $\mathbf{p}'$ is in the cell if the bundles in $D_u(\mathbf{p})$ are *weakly* preferred to all others at this price; that is, we weaken the strict inequality above to a weak inequality (while maintaining the indifference between bundles in $D_u(\mathbf{p})$).[14] So a cell is the intersection of a finite number of half-spaces (sets $\{\mathbf{p}' \in \mathbb{R}^n \mid \mathbf{p}'.\mathbf{v} \leq \alpha\}$ for some $\mathbf{v} \in \mathbb{R}^n$ and some $\alpha \in \mathbb{R}$). Thus:

III Each cell is a closed convex polyhedral set in $\mathbb{R}^n$.

[11] Recall that an affine space in $\mathbb{R}^n$ is a parallel shift of a linear subspace, that is, a set $\{\mathbf{v}+\mathbf{c} \mid \mathbf{v} \in U\}$ for some linear subspace $U \leq \mathbb{R}^n$ and some fixed vector $\mathbf{c}$.

[12] See the equations for the three objects, given below. One might strictly refer to the 'cell interior' as the *relative interior* of the cell.

[13] To be precise, the dimension of a cell is the dimension of its affine span.

[14] It follows that we could alternatively define a cell as those points $\mathbf{p}'$ such that $D_u(\mathbf{p}) \subseteq D_u(\mathbf{p}')$ for some demand set $D_u(\mathbf{p})$.

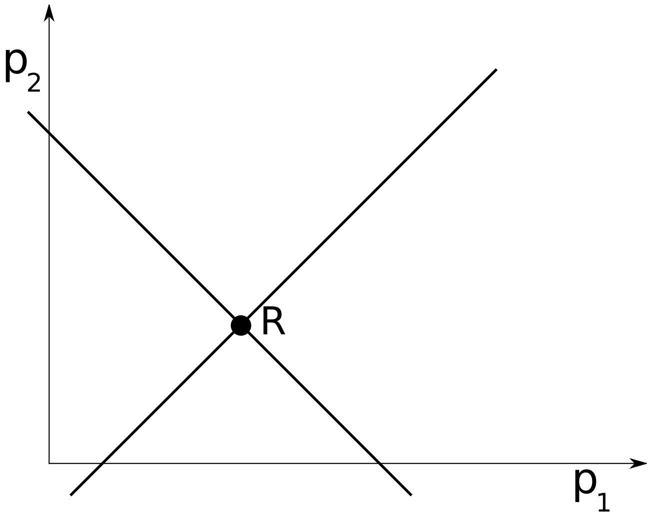


Figure 2: Cell interiors do not intersect; the line segments on either side of R are distinct cells.

The affine span of the cell corresponding to $D_u(\mathbf{p})$ is simply those $\mathbf{p}'$ such that $\mathbf{p}'.(\mathbf{x}-\mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$. So the affine span of the cell is parallel to a linear subspace of $\mathbb{R}^n$, and, since $\mathbf{x}, \mathbf{x}' \in \mathbb{Z}^n$, we have:

IV The slope of the affine span of each cell is rational.

Finally, the boundary of the cell corresponding to $D_u(\mathbf{p})$ is those $\mathbf{p}'$ such that at least one of the weak inequalities $\mathbf{p}'.(\mathbf{x}-\mathbf{x}'') \leq u(\mathbf{x}) - u(\mathbf{x}'')$ for $\mathbf{x} \in D_u(\mathbf{p})$, $\mathbf{x}'' \in A \backslash D_u(\mathbf{p})$ holds with equality. Such points therefore also lie in a lower dimensional cell, so by restricting a suitable choice of inequalities to be equalities, we have:

V The boundary of a k-cell is a union of a finite number of $(k-1)$-cells.

On the other hand, any $(k-1)$-cell lies in the boundary of some k-cell (since, from the equations defining any $(k-1)$-cell, we can obtain the equations defining some k-cell by weakening one or more of the equalities). It follows that a TH is contained in the union of its facets.

We can therefore conclude that every TH for demand over n distinct goods can be understood as an $(n-1)$-dimensional *rational polyhedral complex*:

Definition 2.2 (Mikhalkin, 2004, Definitions 1 and 2)**.** A subset $\Pi \subsetneq \mathbb{R}^n$ is a *rational polyhedral complex* if it is a finite union of closed sets in $\mathbb{R}^n$ called cells which satisfy properties **I**-**V** above. Π is *k-dimensional* if it is contained in the union of its k-cells.

By definition, demand in the complement of a TH is unique. We call a connected component of the complement of a TH a *unique demand region* (UDR). Demand is constant in each UDR, since the bundle demanded cannot change without the price crossing the TH. But to understand how demand changes as we move between UDRs, we need one additional type of information: 'weightings' on the facets.

Let F be a facet and let bundles $\mathbf{x}$ and $\mathbf{x}'$ be demanded in the UDRs on either side. So at prices $\mathbf{p} \in F$, the agent is indifferent between $\mathbf{x}$ and $\mathbf{x}'$, that is, $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} =$

$u(\mathbf{x}') - \mathbf{p}.\mathbf{x}'$. The crucial point is that because $\mathbf{p}.(\mathbf{x}' - \mathbf{x})$ is therefore a constant for these prices, the vector $\mathbf{x}' - \mathbf{x}$ is normal to F. Call the greatest common divisor of the entries of $\mathbf{x}' - \mathbf{x}$ the *weight* of the facet, $w(F)$. So $\mathbf{v}_F := \frac{1}{w(F)}(\mathbf{x}' - \mathbf{x})$ is a primitive integer vector (that is, the greatest common divisor of its entries is 1), and it points from the UDR where $\mathbf{x}'$ is demanded to the UDR where $\mathbf{x}$ is. But since F is $(n-1)$ dimensional, its normal direction is unique, so there is a unique primitive integer normal vector pointing from the UDR of $\mathbf{x}'$ to that of $\mathbf{x}$. Thus knowing only F, $w(F)$ and $\mathbf{x}$ allows us to derive $\mathbf{v}_F$, and hence $\mathbf{x}'$. It therefore follows that if we know demand in any one UDR, we can find demand everywhere from knowing the set of facets (and hence their primitive integer normal vectors) and their weights.

A rational polyhedral complex is described as *weighted* if a positive integer weight is attached to each facet. We provide examples in Section 2.4.

We can now provide the full definition of a TH:

Definition 2.3 (Mikhalkin, 2004, Example 2)**.** Let $A \subsetneq \mathbb{Z}^n$ be a finite set and let $u : A \to \mathbb{R}$ be any function. Then the *tropical hypersurface* $\mathcal{T}_u$ associated with u is the weighted rational polyhedral complex such that:

1. its underlying set is $\{\mathbf{p} \in \mathbb{R} \mid \#D_u(\mathbf{p}) > 1\}$;
2. if F is a facet separating the UDRs U and U', in which $\mathbf{x}$ and $\mathbf{x}'$ are demanded respectively, and if $\mathbf{v}_F$ is the primitive integer vector normal to F which points from U' to U, then the weight of F is the integer $w(F)$ such that $w(F)\mathbf{v}_F = \mathbf{x}' - \mathbf{x}$.

We will see that the TH captures all the information we might ever need to know about an agent's demand and valuation function, if the latter is concave in the standard sense:

Definition 2.4. A function $u : A \to \mathbb{R}$ is *concave* if $A \subsetneq \mathbb{Z}^n$ is convex (as a subset of $\mathbb{Z}^n$) and if u can be extended to a weakly concave function on $\mathbb{R}^n$.

It is a standard result that concave functions are precisely those for which there are no bundles in A that are never demanded. That is:

Lemma 2.5 (cf. e.g. Milgrom and Strulovici, 2009, Theorem 1)**.** *Let $A \subsetneq \mathbb{Z}^n$. A function $u : A \to \mathbb{R}$ is concave iff, for all $\mathbf{x} \in A$, there exists $\mathbf{p} \in \mathbb{R}^n$ such that $\mathbf{x} \in D_u(\mathbf{p})$.*

2.3 Associating demand with geometric objects

When does a weighted rational polyhedral complex depict a valid demand of some agent?

If we construct a TH by starting from some valuation function u, then the weights we attach will necessarily be coherent, since if we cross facets by passing through a sequence of UDRs that ends where it started, we must demand at the end precisely what we demanded at the beginning. In particular, the TH will satisfy the *balancing condition*:

Definition 2.6 (Mikhalkin, 2004, Definition 3)**.** An $(n-1)$-dimensional weighted rational polyhedral complex $\Pi \subsetneq \mathbb{R}^n$ is *balanced* if for every for every $(n-2)$-cell $G \subsetneq \Pi$, the weights $w(F_j)$ on the facets $F_1, \ldots, F_l$ that are adjacent to G, and primitive integer normal vectors $\mathbf{v}_{F_j}$ for these facets that are defined by a rotational direction about G, satisfy $\sum_{j=1}^{l} w(F_j)\mathbf{v}_{F_j} = 0$.[15]

Note that there do not necessarily exist weights to balance a general rational polyhedral complex.[16] However, the balancing condition is in fact the *only* condition that a weighted rational polyhedral complex has to satisfy to be the TH of some valuation function:

Theorem 2.7 (Mikhalkin, 2004, Proposition 2.4; also Mikhalkin, 2005, Theorem 3.15)**.** *Suppose that $\Pi \subsetneq \mathbb{R}^n$ is an $(n-1)$-dimensional balanced weighted rational polyhedral complex.*[17] *Then there exists a finite set $A \subsetneq \mathbb{Z}^n$ and a function $u : A \to \mathbb{R}$ such that Π is the TH, $\mathcal{T}_u$.*

The correspondence between a TH and its associated set A and function u is not unique, but the ambiguities are trivial if u is concave. Clearly, adding a constant to $u(\mathbf{x})$ leaves the TH unchanged, as does increasing every available bundle by a fixed bundle and making a corresponding shift in the valuation (though the bundle demanded in each UDR will then also be increased by the fixed bundle). That is, if $A' = \{\mathbf{x} + \mathbf{c} \mid \mathbf{x} \in A\}$ and $u'(\mathbf{x} + \mathbf{c}) = u(\mathbf{x}) + \alpha$ for all $\mathbf{x} \in A$, some $\mathbf{c} \in \mathbb{Z}^n$, and some $\alpha \in \mathbb{R}$, then $\mathcal{T}_{u'} = \mathcal{T}_u$.

Furthermore, any non-concave u has the same TH as the minimal weakly-concave function that weakly exceeds it everywhere on A. To see this, observe that if a bundle is never demanded then its precise value to the agent is immaterial, so we can increase its value up to the threshold at which it is just marginally demanded for some price(s) without altering the shape or properties of the TH. Doing this for all never-demanded bundles removes any non-concavities in the valuation function. It is also now clear that if two agents have valuations u and u', respectively on different sets of bundles A and A', but their convex hulls in $\mathbb{R}^n$, which we write $\operatorname{Conv} A$ and $\operatorname{Conv} A'$, coincide; and if $\widehat{u}$ is the minimal concave function on $\operatorname{Conv} A$ such that $\widehat{u} \geq u$ on A, and is also the minimal concave function on $\operatorname{Conv} A$ such that $\widehat{u} \geq u'$ on A'; then $\mathcal{T}_{\widehat{u}} = \mathcal{T}_u = \mathcal{T}_{u'}$.[18]

Summing up:

Theorem 2.8 (Mikhalkin, 2004, Remark 2.3)**.** *There is a 1-1 correspondence between THs with an identified 'demand 0' UDR, and pairs (u, A), where $A \subsetneq \mathbb{Z}^n$ is finite and convex in $\mathbb{Z}^n$, u is a weakly-concave, function u on A, for which $u(0) = 0$ and 0 is demanded where specified.*

[15] To choose a rotational direction around G, pick a 2-dimensional affine subspace H of $\mathbb{R}^n$ orthogonal to G, such that the intersection of each F_j with H is 1-dimensional. The intersection of H with the TH is then a collection of 1-cells meeting at the 0-cell which is $G \cap H$. An ordinary choice of rotational direction in this two-dimensional picture gives a rotational direction around G in $\mathbb{R}^n$.

[16] For example, in two dimensions, consider three 0-cells, each with three adjacent facets. If each pair of 0-cells has an adjacent facet in common, the six weights must satisfy six balancing conditions (that is, three equations in each of the two dimensions). But since the balancing conditions are trivially satisfied by setting all weights equal to zero, the conditions can only be satisfied by positive integer weights if the conditions are not linearly independent–which is non-generic.

[17] Strictly speaking, of course, Π is a subset of the space $\mathbb{R}^n$ *and* has weights. As before, we follow Mikhalkin and the mathematical literature in our presentation.

[18] We defined $\widehat{u}$ on $\operatorname{Conv} A \subsetneq \mathbb{R}^n$, but it still defines a TH if it is restricted to $\operatorname{Conv} A \cap \mathbb{Z}^n$.

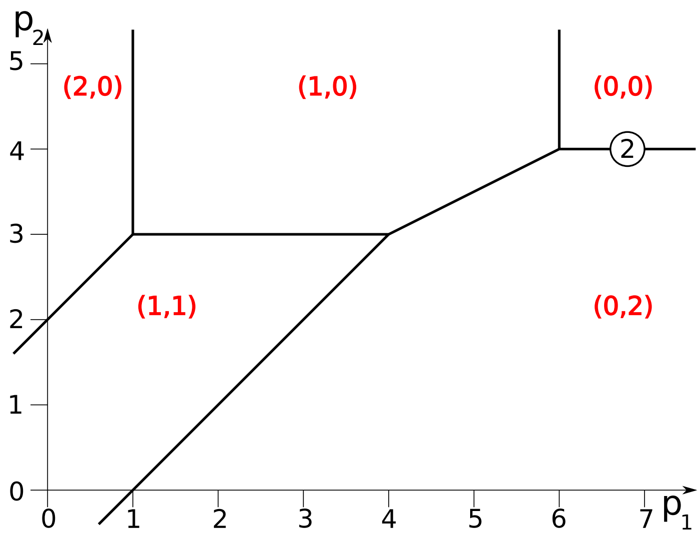


Figure 3: The TH of Example 2.9, with the bundle demanded in each UDR marked in red.

Thus we have full equivalence between THs and weakly-concave valuation functions (such that $u(0) = 0$ and 0 is demanded in a specified UDR). Note, in particular, that a given set in $\mathbb{R}^n$ is the TH of some quasilinear demand if and only if it is a rational polyhedral complex and there exist weights for the facets such that it is balanced. Although we do *not* restrict attention to concave valuation functions–indeed Section 5.2 will ask when an aggregate valuation is concave–understanding of the concave case is important.

Similarly, we do *not* restrict attention to what is demanded in UDRs, but doing so is an important first step. Generically all prices are in a UDR so, as noted above Definition 2.3, given any TH and a specified 'zero demand' UDR we can easily work out what is demanded for a generic price. And it is particularly straightforward to relate properties of demand such as substitutes or complements to the geometry of the TH; see Section 4.

2.4 Demand examples

Example 2.9. Let $A = \{\mathbf{x} \in \mathbb{Z}^2_{\geq 0} \mid x_1 + x_2 \leq 2\}$ and let $u : A \to \mathbb{R}$ be as follows (we arrange the terms in this "back-to-front" way to correspond to the fact that *smaller* quantities will appear *higher* in, and further *right* in, the TH; this convention will be particularly helpful later):

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
7	6	0	$x_2 = 0$
	9	4	$x_2 = 1$
		8	$x_2 = 2$

.

The TH associated with the agent's valuation, u, is shown in Figure 3, in which we have additionally marked in red the bundle demanded by this agent in each UDR. The facet between the UDRs in which $(0, 0)$ and $(0, 2)$ are demanded has weight 2. For $\mathbf{p}$

in this facet (that is, for $p_2 = 4$ and $p_1 > 6$) we have $D_u(\mathbf{p}) = \{(0,0),(0,1),(0,2)\}$; in particular the bundle $(0,1)$ *is* demanded for some price and the function *is* concave. An otherwise-identical valuation u' in which $u'(0,1) < 4$ would give rise to the *same* TH, but would not be concave; $(0,1)$ would not be demanded for any price.

It is easy to work out from the TH which bundle is demanded in each UDR, if one already knows what is demanded in any one UDR. If $x_1 = x_2 = 0$ in the top right UDR we can simply "walk around" the diagram, adding to x_1 (x_2) the weight of any facet crossed times the first (second) coordinate of the primitive integer facet normal. Thus starting from the top right UDR, crossing the vertical facet with normal $(1,0)$, that is, $\{\mathbf{p} \in \mathbb{R}^2 \mid p_1 = 6, p_2 > 4\}$, changes demand from $(0,0)$ to $(1,0)$; from there, crossing the facet with normal $(-1,2)$ changes demand to $(0,2)$, as may also be seen by crossing the weight-2 horizontal from $(0,0)$ downwards; and so on.

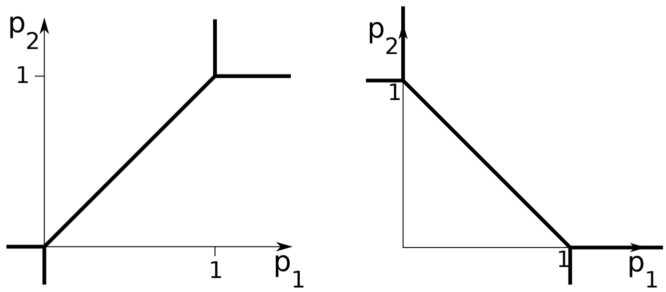
Example 2.10. It will be useful later to discuss very simple examples of substitutes and complements demands: if $A = \{0,1\}^2$, then $u^1 : A \to \mathbb{R}$ and $u^2 : A \to \mathbb{R}$ defined as follows are demands for substitutes and complements, respectively, and their THs are shown in Figures 4a and 4b.[19]

$x_1 = 1$	$x_1 = 0$	u^1
1	0	$x_2 = 0$
1	1	$x_2 = 1$

and

$x_1 = 1$	$x_1 = 0$	u^2
0	0	$x_2 = 0$
1	0	$x_2 = 1$

.



(a) $\mathcal{T}_{u^1}$. (b) $\mathcal{T}_{u^2}$.

Figure 4: The THs of Example 2.10.

Clearly each TH has four UDRs in which these agents demand the bundles $(0,0)$, $(0,1)$, $(1,1)$, and $(1,0)$, respectively, as one moves clockwise around the UDRs starting at the top right–as is also easily confirmed by adding the appropriate primitive integer facet normal on every crossing between UDRs.

Example 2.11. For a simple 3-dimensional example, let $A = \{\mathbf{x} \in \mathbb{Z}^3_{\geq 0} \mid x_1 + x_2 + x_3 \leq 1\}$ and let $u(0,0,0) = 0$ and $u(1,0,0) = u(0,1,0) = u(0,0,1) = 1$. The TH is given in Figure 5. Now, the facets are 2-dimensional (pieces of planes), there are additionally 1-cells (lines along which these facets meet), and a 0-cell (point at which these lines meet). Three of these facets, having normals $(1,0,0)$ (dark-green facet), $(0,1,0)$ (red

[19]The TH of Figure 4a appears to be a translation of Figure 1, but there is an important distinction. In Figure 1 the support is $\{(0,0),(0,1),(1,0)\}$, so the TH has only one 0-cell; here, u^1 has support $\{0,1\}^2$, and its TH has two 0-cells. (If we restricted u^1 to the support $\{(0,0),(0,1),(1,0)\}$ its TH would coincide with $\mathcal{T}_{u^1}$ on $\mathbb{R}^2_{\geq 0}$ but have only one 0-cell.)

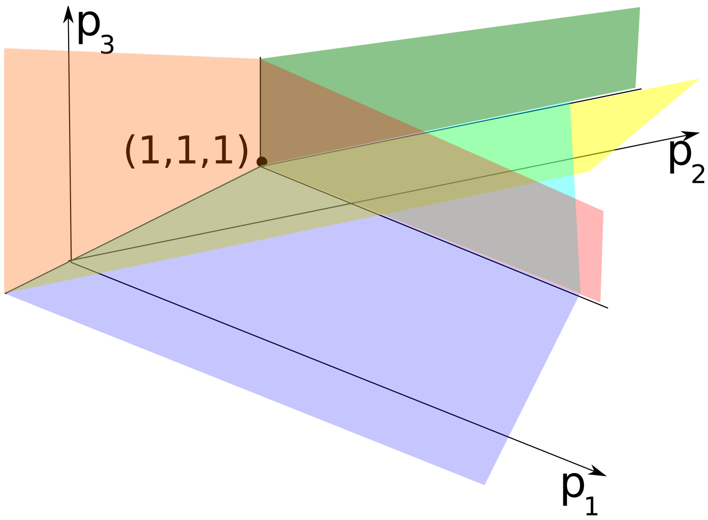


Figure 5: The TH of Example 2.11.

facet), and $(0,0,1)$ (turquoise facet), border the UDR in which $(0,0,0)$ is demanded; this UDR is of course $\{\mathbf{p} \in \mathbb{R}^3 \mid p_1, p_2, p_3 > 1\}$. Crossing any one of these facets takes us to the UDR in which the corresponding bundle is demanded. We swap between the latter UDRs by crossing the remaining three facets, which have normals $(1,-1,0)$ (orange facet), $(0,1,-1)$ (bluish-purple facet) and $(1,0,-1)$ (yellow facet).

3 Duality in Tropical Geometry

The previous section demonstrated the equivalence between THs and specific valuation functions. However, we now describe a coarser correspondence between a *set* of THs that are "essentially" the same as one another, and *sets* of valuation functions which–we will see–have the same fundamental properties.

Looking, e.g., at Figure 1, the important structure is that there are particular UDRs and particular sets of prices at which the agent is indifferent between the bundles of those UDRs. So we say that two THs have the same *combinatorial type* if there is a 1-1 correspondence between the cells of the THs which have the same dimension and slope, and these cells connect to one another in the same way. Demands corresponding to THs of the same combinatorial type are "essentially" the same in that they represent agents who make the same trade-offs between additional units of goods, even if not always at the same prices. We will show that all THs of the same combinatorial type are, in a precise way, dual to a particular subdivision of Conv A.

3.1 Duality between convex polytopes and cells

Although we assume that goods are indivisible, we now develop a structure of convex polytopes and their faces in quantity space, so extend our focus from A to Conv $A \subsetneq \mathbb{R}^n$.[20] We first show that this extension has no effect on the way we separate prices into

[20]Definitions and basic properties of standard geometric terms (polytopes, polyhedral sets, faces, etc.) are in Appendix A.

different cell interiors by showing that $\operatorname{Conv} D_u(\mathbf{p}) = \operatorname{Conv} D_u(\mathbf{p}') \iff D_u(\mathbf{p}) = D_u(\mathbf{p}')$, for any prices $\mathbf{p}$ and $\mathbf{p}'$. To do this we first prove a lemma, the Pseudo-equilibrium Prices Lemma, that will be very important in the sequel:

Lemma 3.1 (Pseudo-equilibrium Prices Lemma, Milgrom and Strulovici, 2009, Proposition 2)**.** *Let u be **any** valuation function. Suppose $\mathbf{p}$ is any price vector, and $\mathbf{x}$ is an integer bundle in $\operatorname{Conv} D_u(\mathbf{p})$. If there exists any price vector $\mathbf{p}'$ such that $\mathbf{x} \in D_u(\mathbf{p}')$, then $\mathbf{x} \in D_u(\mathbf{p})$.*

Proof. For all $\mathbf{x}^\beta \in D_u(\mathbf{p})$, we know $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} \leq u(\mathbf{x}^\beta) - \mathbf{p}.\mathbf{x}^\beta$, with equality only if $\mathbf{x} \in D_u(\mathbf{p})$. So if $\mathbf{x} \in \operatorname{Conv} D_u(\mathbf{p})$, i.e., $\mathbf{x} = \sum_\beta \lambda_\beta \mathbf{x}^\beta$ for some $\lambda_\beta \in [0,1]$ with $\sum_\beta \lambda_\beta = 1$, then it follows that $u(\mathbf{x}) - \mathbf{p}.\mathbf{x} = \sum_\beta \lambda_\beta (u(\mathbf{x}) - \mathbf{p}.\mathbf{x}) \leq \sum_\beta \lambda_\beta u(\mathbf{x}^\beta) - \sum_\beta \lambda_\beta \mathbf{p}.\mathbf{x}^\beta = \sum_\beta \lambda_\beta u(\mathbf{x}^\beta) - \mathbf{p}.\mathbf{x}$ and so, simplifying, that $u(\mathbf{x}) \leq \sum_\beta \lambda_\beta u(\mathbf{x}^\beta)$, with equality only if $\mathbf{x} \in D_u(\mathbf{p})$.

Now suppose $\mathbf{x} \in D_u(\mathbf{p}')$. Then $u(\mathbf{x}) - \mathbf{p}'.\mathbf{x} \geq u(\mathbf{x}^\beta) - \mathbf{p}'.\mathbf{x}^\beta$ for all $\mathbf{x}^\beta$ so we similarly show that $u(\mathbf{x}) \geq \sum_\beta \lambda_\beta u(\mathbf{x}^\beta)$. Hence, if $\mathbf{x} \in D_u(\mathbf{p}')$ for any $\mathbf{p}'$, then $\mathbf{x} \in D_u(\mathbf{p})$. $\square$

Note that we do not require that u is necessarily concave.

Corollary 3.2. *For **any** valuation function, u, if $\mathbf{p}$ and $\mathbf{p}'$ are any two price vectors, then $\operatorname{Conv} D_u(\mathbf{p}) = \operatorname{Conv} D_u(\mathbf{p}') \iff D_u(\mathbf{p}) = D_u(\mathbf{p}')$.*

Proof. It is immediate from Lemma 3.1 that if $\mathbf{x} \in D_u(\mathbf{p}') \subseteq \operatorname{Conv} D_u(\mathbf{p}') = \operatorname{Conv} D_u(\mathbf{p})$ then $\mathbf{x} \in D_u(\mathbf{p})$, so the result follows. $\square$

For any price, $\mathbf{p}$, we write $\Delta(\mathbf{p}) := \operatorname{Conv} D_u(\mathbf{p})$ for this polytope in (divisible) quantity space $\mathbb{R}^n$. From Definition 2.1, and Corollary 3.2 we can write the associated cell interior as $\{\mathbf{p}'' \in \mathbb{R}^n \mid \Delta(\mathbf{p}) = \Delta(\mathbf{p}'')\}$, and since it is therefore defined by the polytope $\Delta(\mathbf{p})$, we write $C_{\Delta(\mathbf{p})}$ for the corresponding cell (its closure). Recall from the discussion in Section 2.2 that a price $\mathbf{p}''$ is in the cell $C_{\Delta(\mathbf{p})}$ iff the bundles in $D_u(\mathbf{p})$ are weakly preferred to all others at price $\mathbf{p}''$, i.e., iff $D_u(\mathbf{p}) \subseteq D_u(\mathbf{p}'')$.[21] Applying Corollary 3.2 again, we conclude that $C_{\Delta(\mathbf{p})} = \{\mathbf{p}'' \in \mathbb{R}^n \mid \Delta(\mathbf{p}) \subseteq \Delta(\mathbf{p}'')\}$. It follows immediately

$$\Delta(\mathbf{p}) \subsetneq \Delta(\mathbf{p}') \iff C_{\Delta(\mathbf{p}')} \subsetneq C_{\Delta(\mathbf{p})}. \tag{2}$$

We now describe the dualities between the polytope $\Delta(\mathbf{p})$ in quantity space, and the associated cell $C_\Delta(\mathbf{p})$ in price space; we show how they extend to the global structure in the next subsection.

First, note the dimensions of $\Delta(\mathbf{p})$ and $C_{\Delta(\mathbf{p})}$ are dual. $C_{\Delta(\mathbf{p})}$ has the dimension of its affine span, that is, of that set of prices $\mathbf{p}'$ such that $\mathbf{p}'.(\mathbf{x}-\mathbf{x}') = u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$. If $\Delta(\mathbf{p})$ is k-dimensional, these equations impose k linearly independent constraints on such $\mathbf{p}'$, so $\dim C_{\Delta(\mathbf{p})} = n - k$.

Next observe the affine spans of these sets are orthogonal: since $\mathbf{p}'.(\mathbf{x}-\mathbf{x}')$ is constant for all $\mathbf{p}' \in C_{\Delta(\mathbf{p})}$ and all $\mathbf{x}, \mathbf{x}' \in D_u(\mathbf{p})$, we have $(\mathbf{p}' - \mathbf{p}'').(\mathbf{x} - \mathbf{x}') = 0$ for any $\mathbf{p}', \mathbf{p}'' \in C_{\Delta(\mathbf{p})}$ and $\mathbf{x}, \mathbf{x}' \in \Delta(\mathbf{p})$. So all prices in $C_{\Delta(\mathbf{p})}$ lie in a subspace of $\mathbb{R}^n$ orthogonal to any $\mathbf{x} - \mathbf{x}'$ where $\mathbf{x}, \mathbf{x}' \in \Delta(\mathbf{p})$, and all bundles in $\Delta(\mathbf{p})$ lie in a subspace of $\mathbb{R}^n$ orthogonal to $\mathbf{p}' - \mathbf{p}''$ for any $\mathbf{p}', \mathbf{p}'' \in C_{\Delta(\mathbf{p})}$.

[21] See also Footnote 14.

Therefore, any ($n-1$ dimensional) facet $F = C_{\Delta(\mathbf{p})}$ (in price space) corresponds to a 1-dimensional polytope, i.e., a line-segment, $\Delta(\mathbf{p})$, orthogonal to it (in quantity space). And if $\mathbf{x}$ and $\mathbf{x}'$ are the endpoints of the line-segment $\Delta(\mathbf{p})$, then $\mathbf{x}-\mathbf{x}' = w\mathbf{v}_F$ for some $w \in \mathbb{Z}$, where $\mathbf{v}_F$ is a primitive integer vector in the direction of $\Delta(p)$, i.e. in the normal direction to F; let us chose $\mathbf{v}_F$ so that $w > 0$. And since the bundles demanded in the UDRs on either side of F are precisely the vertices at the endpoints of $\Delta(\mathbf{p})$, it also follows that this w is the weight of F, as defined in Section 2.2. In words, the "length" of the line-segment $\Delta(\mathbf{p})$ in quantity space is the weight of its corresponding facet in price space.

3.2 The subdivided Newton Polytope

Convex geometry now provides a clever trick to find the set of all the polytopes, $\Delta(\mathbf{p})$, very quickly, and see how they fit together in quantity space. From this it is easy to see how the cells of the TH fit together in price space.

The condition that a bundle $\mathbf{x}' \in D_u(\mathbf{p})$ maximises the agent's surplus at price $\mathbf{p}$ can be re-written using vectors in $\mathbb{R}^{n+1}$ as $(-\mathbf{p}, 1).(\mathbf{x}, u(\mathbf{x})) \leq (-\mathbf{p}, 1).(\mathbf{x}', u(\mathbf{x}'))$ for all $\mathbf{x} \in A$. So the points $(\mathbf{x}, u(\mathbf{x}))$, for all $\mathbf{x} \in A$, must lie in a particular half-space of $\mathbb{R}^{n+1}$. Furthermore all the other bundles $\mathbf{x}''$ which are optimal at the same price $\mathbf{p}$ satisfy $(-\mathbf{p}, 1).(\mathbf{x}'', u(\mathbf{x}'')) = (-\mathbf{p}, 1).(\mathbf{x}', u(\mathbf{x}'))$ and so all lie on the hyperplane in $\mathbb{R}^{n+1}$ bounding this half-space. Hence every set $\Delta(\mathbf{p})(= \mathrm{Conv}\, D_u(\mathbf{p}))$ is the projection to the first n coordinates of a face of the set

$$\widehat{A} := \mathrm{Conv}\{(\mathbf{x}, u(\mathbf{x})) \in \mathbb{R}^{n+1} \mid \mathbf{x} \in A\}. \tag{3}$$

Conversely, consider any face $\widehat{\Delta}$ of $\widehat{A}$ on the 'upper side' with respect to the final coordinate (i.e., any face such that points with a slightly lower final coordinate than those in the face are in $\widehat{A}$, and those with a slightly higher final coordinate are not). $\widehat{\Delta}$ is the intersection of $\widehat{A}$ with some hyperplane $\{\mathbf{y} \in \mathbb{R}^{n+1} \mid \mathbf{y}.\mathbf{v} = \alpha\}$ for some $\alpha \in \mathbb{R}$, and some normal vector $\mathbf{v} \in \mathbb{R}^{n+1}$. We know $\widehat{A}$ is contained in the half-space below the hyperplane with respect to the final coordinate. Renormalising so the final coordinate of $\mathbf{v}$ is 1, so $\mathbf{v} = (-\mathbf{p}, 1)$ for some vector $\mathbf{p} \in \mathbb{R}^n$, the face $\widehat{\Delta}$ is the convex hull of all points $(\mathbf{x}', u(\mathbf{x}'))$, where $\mathbf{x}'$ is in A, satisfying $(-\mathbf{p}, 1).(\mathbf{x}, u(\mathbf{x})) \leq (-\mathbf{p}, 1).(\mathbf{x}', u(\mathbf{x}'))$ for all $\mathbf{x} \in A$; that is, $u(\mathbf{x}') - \mathbf{p}.\mathbf{x}'$ is maximal over bundles in A. Thus the projection of $\widehat{\Delta}$ to its first n coordinates is exactly $\Delta(\mathbf{p})$ for this $\mathbf{p}$.

Summarising, each upper face of $\widehat{A}$ is the set $(\mathbf{x}, u(\mathbf{x}))$ that are maximal when viewed in the direction of some vector $(-\mathbf{p}, 1)$; the face then projects to $\Delta(\mathbf{p})$. And conversely, any set $\Delta(\mathbf{p})$ is the projection of an upper face of $\widehat{A}$. So the information about the demand sets is contained in the projections of these faces, that is, in the collection of sets $\left\{\mathbf{x} \mid (\mathbf{x}, u(\mathbf{x})) \in \widehat{\Delta}\right\}$, where $\widehat{\Delta}$ is an upper face of $\widehat{A}$.

Definition 3.3.

1. The subdivision of $\mathrm{Conv}\, A$ given by the projections of the upper faces of $\widehat{A}$ onto $\mathrm{Conv}\, A$ is a *subdivided Newton polytope* (SNP).[22]

[22]It is a subdivision of the set $\mathrm{Conv}\, A$ which is itself called a Newton polytope in (tropical) algebraic geometry.

2. The image Δ of a k-dimensional face $\widehat{\Delta}$ of $\widehat{A}$ is a *k-face* of the SNP.

We give an example of how to construct an SNP in practice in Section 3.3.

Since, for $k < n$, any k-face of $\widehat{A}$ is the face of an n-face of $\widehat{A}$, it is sufficient to consider only the maximal faces of $\widehat{A}$ to identify the full SNP structure.

In particular, an SNP n-face, Δ, is the projection of an upper n-face $\widehat{\Delta}$ of $\widehat{A}$. But since $\widehat{\Delta}$ is n-dimensional, there is a unique hyperplane of $\mathbb{R}^{n+1}$ passing through it, and so a unique normal vector of the form $(-\mathbf{p}, 1)$. So the projection Δ of $\widehat{\Delta}$ to $\operatorname{Conv} A$ is exactly $\Delta(\mathbf{p}) = \operatorname{Conv} D_U(\mathbf{p})$, and is *not* $\Delta(\mathbf{p}')$ for any $\mathbf{p}' \neq \mathbf{p}$. So $\mathbf{p}$ is the *only* price at which *all* these bundles are demanded, and $\{\mathbf{p}\}$ is therefore a 0-cell in the TH, i.e. $\{\mathbf{p}\} = C_{\Delta(\mathbf{p})}$.

At the other extreme, for any 0-face $\{\mathbf{x}\}$ of the SNP, there exist prices $\mathbf{p}$ at which $(\mathbf{x}, u(\mathbf{x}))$ is the unique point of $\widehat{A}$ intersecting a supporting hyperplane normal to $(-\mathbf{p}, 1)$. For any such $\mathbf{p}$ we know $\{\mathbf{x}\} = D_u(\mathbf{p})$. Furthermore, if any such $\mathbf{p}$ is changed infinitesimally in any coordinate direction, the point $\{(\mathbf{x}, u(\mathbf{x}))\}$ is still the unique point of $\widehat{A}$ intersecting the corresponding supporting hyperplane. So the UDR in which $\mathbf{x}$ is demanded, that is, the set of $\mathbf{p}$ such that $\{\mathbf{x}\} = D_u(\mathbf{p})$, is (of course) n-dimensional.

Between these extremes, any upper k-face of $\widehat{A}$, where $2 \leq k \leq n-1$, is the intersection of $\widehat{A}$ with any one of a range of hyperplanes in $\mathbb{R}^{n+1}$. The vector $(-\mathbf{p}, 1)$ normal to any such hyperplane defines a price $\mathbf{p}$ lying in the corresponding $(n-k)$-dimensional cell interior of the TH.

The fact that the SNP's faces, $\Delta(\mathbf{p})$, are the projections of faces of a convex set tells us how they fit together, and hence how the sets $D_u(\mathbf{p})$ fit together. If $\Delta(\mathbf{p}) \subsetneq \Delta(\mathbf{p}')$ for two faces of the SNP, then $\Delta(\mathbf{p})$ must be a face of the polytope $\Delta(\mathbf{p}')$. But recall (displayed equation 2) that $\Delta(\mathbf{p}) \subsetneq \Delta(\mathbf{p}')$ iff $C_{\Delta(\mathbf{p}')} \subsetneq C_{\Delta(\mathbf{p})}$. As discussed above (at and beneath point **V** of Section 2.2,) the latter holds iff $C_{\Delta(\mathbf{p}')}$ is in the boundary of $C_{\Delta(\mathbf{p})}$. Moreover, $\Delta(\mathbf{p})$ and $C_{\Delta(\mathbf{p})}$ are orthogonal, as discussed in Section 3.1. So knowing how the $\Delta(\mathbf{p})$ fit together in quantity space makes it immediately obvious how the $C_{\Delta(\mathbf{p})}$ fit together in price space, and vice versa.

So the SNP tells us which cells must exist in the corresponding THs, their slopes, and how they are connected. In other words

Theorem 3.4 (Mikhalkin, 2004, Proposition 2.1.)**.** *For a given* $\operatorname{Conv} A$ *there is a 1-1 correspondence between SNPs and combinatorial types of THs.*

As noted above, this correspondence is coarser than the correspondence we described in the previous subsection (Theorem 2.8): different valuations correspond to the same SNP, and hence to a TH of same combinatorial type, even though the coordinates of the parts of the TH differ. However, this correspondence isolates the underlying properties of demands, specifically the sets of bundles one might ever be indifferent between, and the trade-offs one might make.

Also, starting from any SNP, it is easy to find the combinatorial type of the TH, and so see which coordinates uniquely define the TH. The TH can then be completely identified using the valuation u. We illustrate this in Section 3.3.

Another important point follows: if A is small, it is easy to list all the possible SNPs, and hence also list all possible combinatorial types of THs for the set A. That is, we can easily list every possible distinct structure of trade-offs that an agent might make between a given finite collection of goods.

Of course, we do not need to start with the SNP. Given the TH and an identified 'demand 0' UDR, we can easily infer both A and the full SNP using the duality described in this section.

Note, however, that if we do not know *ex ante* whether a TH is concave, then neither the TH nor the SNP can necessarily tell us which bundles are demanded in each cell of the TH. The information we do have is as follows:

Corollary 3.5. *Let A be convex in $\mathbb{Z}^n$, let $u : A \to \mathbb{R}$ be a valuation, and consider the corresponding SNP.*

1. *A bundle $\mathbf{x} \in A$ is a vertex of the SNP iff it is demanded in some UDR of the corresponding TH.*

2. *If every bundle $\mathbf{x} \in A$ is a vertex of the SNP, then $\widehat{u}$ is concave for every valuation $\widehat{u} : A \to \mathbb{R}$ such that $\mathcal{T}_{\widehat{u}} = \mathcal{T}_u$.*

3. *If a bundle $\mathbf{x} \in A$ is not a vertex of the SNP, there exist valuations $\widehat{u} : A \to \mathbb{R}$ such that $\mathcal{T}_{\widehat{u}} = \mathcal{T}_u$ but $\mathbf{x} \notin D_{\widehat{u}}(\mathbf{p})$ for any $\mathbf{p} \in \mathbb{R}$.*

Proof. 1 is clear from the previous discussion. 2 follows from Lemma 2.5. For 3, define $\widehat{u}$ to be equal to u on the vertices of the SNP, and to be arbitrarily large negative numbers on those bundles in A that are not vertices of the SNP. $\square$

3.3 Examples

Example 3.6. Starting from a valuation function, a TH can easily be drawn by first deriving the SNP, then using the SNP to draw the shape of the TH's combinatorial type, and finally using the valuations to fix the TH's exact location in price space.

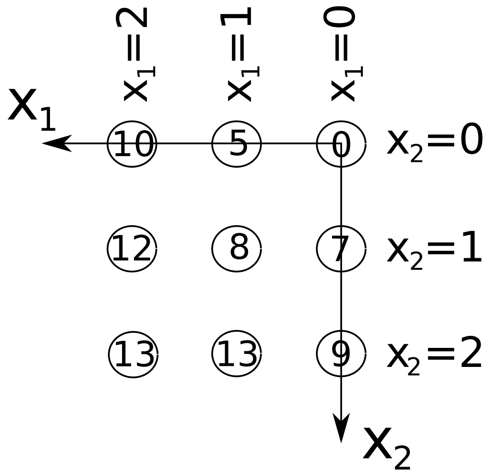
Figure 6 presents a valuation function u, both in the usual tabular representation, and by showing the permissible set of bundles A, as a subset of the lattice $\mathbb{Z}^n$, labelled with their valuations. As before, the quantity of good 1 increases as we move to the *left*, and the quantity of good 2 increases as we move *down*, in order to show the duality between the SNP and the TH most clearly.

Figure 7 adds a third dimension to Figure 6b. Figure 7a shows the points $(\mathbf{x}, u(\mathbf{x}))$ for all $\mathbf{x} \in A$, with the valuations $u(\mathbf{x})$ drawn as lines connecting them to their associated bundles, $\mathbf{x}$, to make the relationships clearer. Figure 7b then pictures the upper surface of $\widehat{A}$, with those lines that correspond to bundles that are demanded for some price(s) in bold. Note that the valuation is non-concave and the bundle $(1, 1)$ is never demanded.

The SNP is pictured in Figure 8. It is drawn *without* axes, since replacing A with $A + \mathbf{c}$ for some $\mathbf{c} \in \mathbb{Z}^n$ and re-defining u to correspond gives us the *same* SNP and TH. A depiction of the SNP and an example of a TH of the corresponding combinatorial type is given in Figure 9, colour-coded so that objects that are the geometric duals of each other have the same colours as each other. That is, each point in the TH has the same colour as its corresponding area in the SNP; each line-segment (facet) in the TH has the same colour as the line-segment (edge) in the SNP that it corresponds, and is *orthogonal* to; and the white areas (UDRs) in the TH correspond to the white points (bundles that are vertices) in the SNP.

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	u
10	5	0	$x_2 = 0$
12	8	7	$x_2 = 1$
13	13	9	$x_2 = 2$

(a) Tabular representation of the valuation.



(b) Each circled number gives the value of the bundle in that position.

Figure 6: Alternative representations of a valuation function.

Note that the black point in the SNP that represents the bundle $(1, 1)$ has no object corresponding to it in the TH–it is "hidden" inside the scarlet-coloured point in the TH. If that bundle's valuation were greater so that, rather than the line corresponding to it in Figure 7b lying strictly below a plane coincident with $\widehat{A}$, the line instead just touched that plane,[23] then the bundle would be demanded at the price corresponding to the scarlet-coloured point in the TH. And if the bundle had a still higher valuation, that point in the TH would "open up" to form an area corresponding to the range of prices at which the bundle would then be demanded.

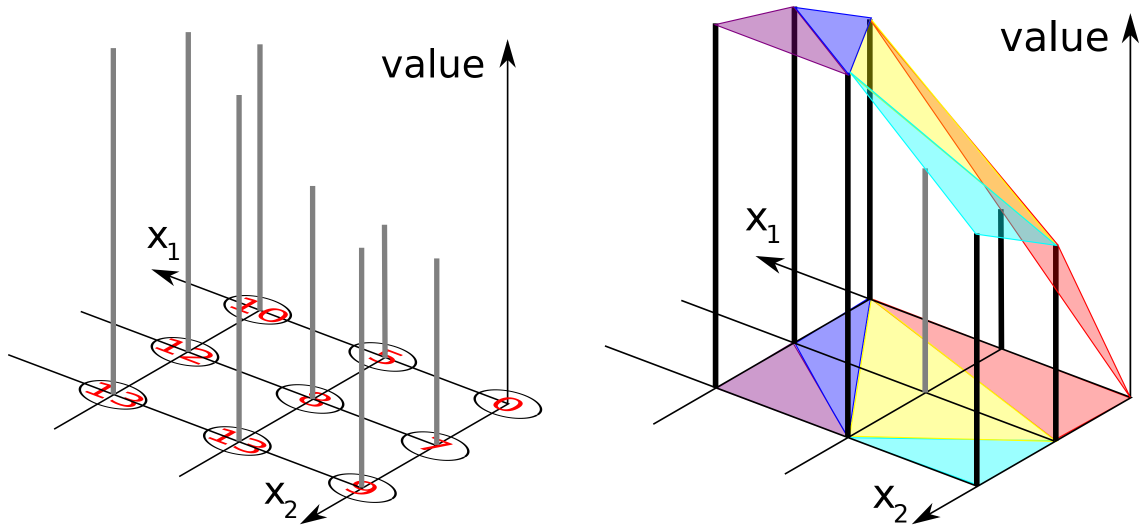
The final SNP lattice point is coloured grey. It is not an SNP vertex, but lies within (horizontal) SNP edge of the same colour, which has "length" 2 (more precisely, the greatest common divisor of the differences (2, and 0) between the co-ordinates of the bundles at the ends of this edge is 2). And this corresponds to the vertical grey facet in the TH which is labelled "2", reflecting its weight.

Finally, remember that Figure 9b shows only one of many THs of the combinatorial type corresponding to the SNP in that figure; the SNP is silent on the lengths of the lines in its corresponding THs. However, the exact location of the TH for our specific set of valuations can easily be worked out from the valuations of different bundles: See Figure 10.

For example, it is clear from the valuations of bundles (1,0) and (0,1) that the top right (pinky-purple) point of the TH is at $\mathbf{p} = (5, 7)$, since 5 and 7 are the prices below which the agent will first buy any of goods 1 and 2, respectively,when the other good's price is very high. And the coordinates of the purple point at the bottom right of the TH must be (4,2) since $9 - 7 = 2$ is the incremental value of a second unit of good 2, when the agent has no unit of good 1, and $13 - 9 = 4$ would be the further increment in value from then also having a unit of good 1, etc.

We discussed above (Section 2.2; see especially Example 2.9) how the demand in each UDR can easily be worked out from the TH.

[23]It is easy to compute that the valuation of this bundle would have to be 10 for this to happen.



(a) The values $(\mathbf{x}, u(\mathbf{x}))$ for all $\mathbf{x} \in A$.

(b) The upper surface of $\widehat{A}$.

Figure 7: Finding $\widehat{A}$ in three dimensions.

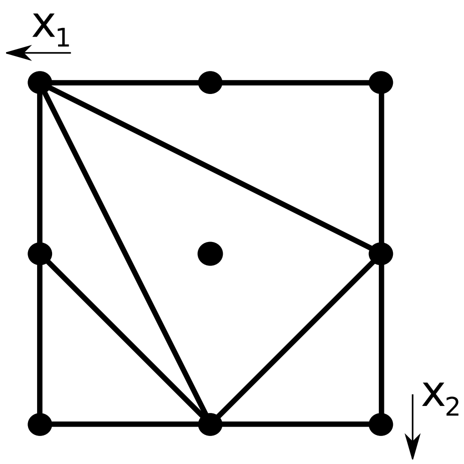


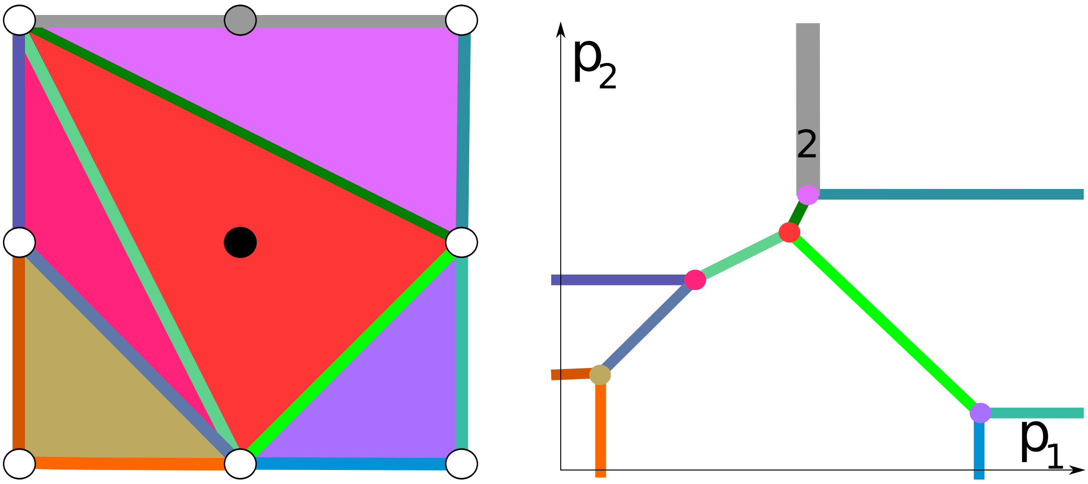
Figure 8: The SNP.

Example 3.7. (Example 2.9 revisited.) It is not hard to check that the SNP for Example 2.9 is as shown in Figure 11a. Two examples of THs of the corresponding combinatorial type are given in Figures 11b and 11c.

Example 3.8. For a fixed A, it is easy to draw every possible SNP and so obtain every possible combinatorial type of TH, thus enumerating all possible "essentially-different" structures of demand. We do this for $A = \{0, 1\}^2$ in Figure 12.

It is not hard to see that Figure 12a applies when $u(0,0) + u(1,1) < u(1,0) + u(0,1)$, so represents substitutes; Figure 12b applies when $u(0,0) + u(1,1) = u(1,0) + u(0,1)$, so is additively separable demand; and Figure 12c applies when $u(0,0) + u(1,1) > u(1,0) + u(0,1)$, so is complements. (Recall Figure 4.) Importantly, it is clear that these are the *only* possibilities.

Observe that Figure 12b can be seen as a limit of Figure 12a (or, equivalently, Figure 12c). In the TH, the two 0-cells become arbitrarily close and then coincide in the limit; in the SNP, the faces of $\widehat{A}$ tilt until they are coplanar when the SNP edge distinguishing



(a) The SNP, colour-coded. (b) A TH, colour-coded to correspond.

Figure 9: The SNP and a TH of the corresponding combinatorial type, colour-coded so that dual geometric objects have the same colours.

them disappears in this limit.

Likewise, any SNP in which the subdivision is not maximal (that is, additional valid $(n-1)$-faces could be added) can be recovered by deleting $(n-1)$-faces from some SNP whose subdivision *is* maximal; the corresponding TH is a limit (or 'degeneration'). Even for larger supports than $A = \{0,1\}^2$, we can efficiently enumerate all those combinatorial types of demand for which the SNP subdivision is maximal, knowing we can recover the remainder as their limits. We do this for $A = \{0,1,2\} \times \{0,1\}$ in Figure 13.

With a bit of practice, starting from either the TH or SNP it is easy to draw the other figure quite fast, at least in two dimensions: if we start with the TH, we know each area around the TH corresponds to a vertex in the SNP, and areas that are separated by a line-segment in the TH correspond to vertices that are connected by a line-segment in the SNP. So we can immediately draw all the vertices and lines. The remaining task is to "straighten out the SNP" without changing it topologically, noting that each line-segment in the SNP is orthogonal to its corresponding line-segment in the TH, and that where a line-segment of weight N is crossed in the TH, there are $(N-1)$ points between the vertices of the corresponding line-segment in the SNP. (The existence of additional points in the SNP that are not on any line segment becomes apparent once the relative positions of all lines are fixed.) Going from the SNP to the TH essentially reverses the process, as we illustrated in Example 3.6, above.

4 Classifying demands

The previous sections suggest classifying demands according to the normal vectors that determine the shapes of agents' THs. We now show that defining demand 'types' in this way does indeed provide a simple characterisation of the standard concepts of substitutes and complements, as well as more recently developed concepts such as strong substitutes, and gross substitutes and complements.

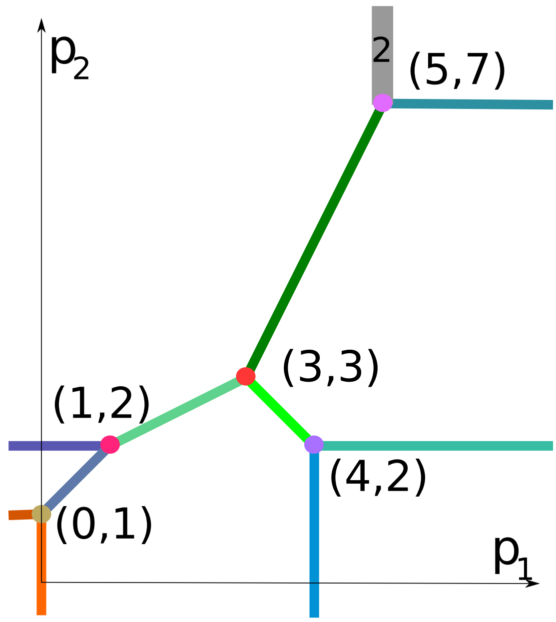


Figure 10: The TH of the valuation function presented in Figure 6.

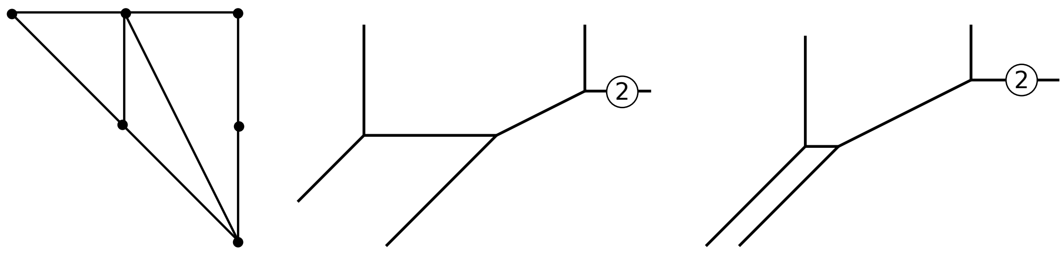


Figure 11: (a) the SNP of Example 2.9; (b) and (c) two examples of THs of the combinatorial type of Example 2.9.

This offers an quick way to check whether demand is, e.g., strong substitutes, since there are easy software solutions to calculate the normal vectors of the TH for any valuation function, u, and hence also immediately yield the demand's 'type'.

Our approach also provides a natural answer to the question of when demand 'types' are effectively equivalent: if and only if they are unimodular basis changes of each other. Furthermore, demand 'types' are a simple framework in which to develop additional distinctions between classes of demands.[24]

Finally, we will show in Section 5 that our framework also allows us to develop new results about aggregate demand, for example, about the existence of competitive equilibrium.

[24]In other work, we use our framework to derive implications about the scope of possible demand functions which are substitutes; for example, various marginal valuations must be equal. See also Fujishige and Yang (2003).

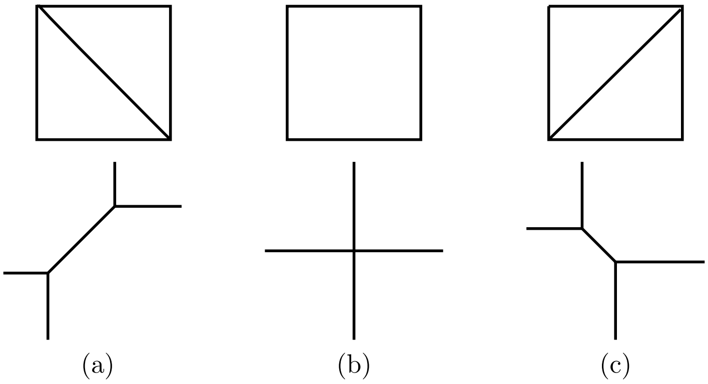


Figure 12: All the possible SNPs, and examples of their corresponding combinatorial types of TH when $A = \{0,1\}^2$.

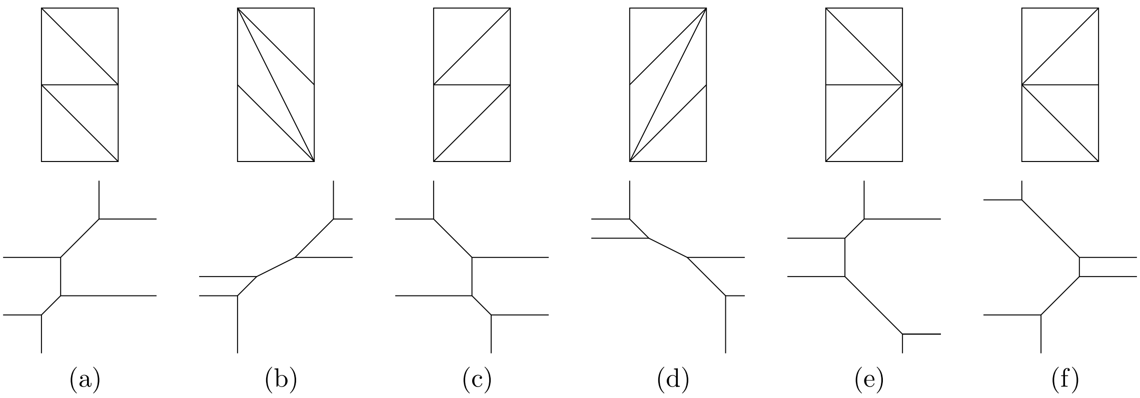


Figure 13: All the possible SNPs with maximal subdivision, and examples of their corresponding combinatorial types of TH, when $A = \{0,1\} \times \{0,1,2\}$.

4.1 Demand Types

Let $\mathcal{D} = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$ be a set of primitive integer vectors in $\mathbb{Z}^n$, such that if $\mathbf{v} \in \mathcal{D}$ then $-\mathbf{v} \in \mathcal{D}$. We will, however, abuse notation by writing the set to include just one representative of each such pair:

Definition 4.1. An agent has *demand of type* $\mathcal{D}= \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$ if all the primitive integer normals to the facets of the TH of its demand lie in $\mathcal{D}$.

An agent has *concave demand of type* $\mathcal{D}$ if its demand is concave and of type $\mathcal{D}$.

It would of course be equivalent to define the demand type of an agent by referring to the edges of its SNP.

We will represent $\mathcal{D}$ by any $n \times r$ matrix D whose columns are the vectors of $\mathcal{D}$. Of course, D is not unique, since its columns can be in any order, whereas the set $\mathcal{D}$ *is* unique.[25] For example, any of a number of matrices including, for example,

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix},$$

represent the demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ of the THs in Figures 1, 4a, 12a, 12b, and 13a.[26] Note that a TH has any demand type which contains its facet normals; we do not restrict to the *minimal* such set. (So, for example, any of the THs listed earlier in this paragraph are *also* of type $\{\mathcal{D}, \pm(N_1, N_2)\}$, for any primitive integer (N_1, N_2).)

It is straightforward that two demands share many properties if one can be transformed into the other by a unimodular basis change.[27] Such a basis change is equivalent to re-packaging the goods so that any integer bundle can still be obtained by buying and selling an (integer) selection of the new packages; and any integer selection of the new packages was available as an integer combination of the original goods. So such a basis change has no "real" effect.

Likewise, such a basis change simply distorts the TH by a linear transformation which leaves its important structure unaffected:

Proposition 4.2. *For $A \subsetneq \mathbb{Z}^n$ and $u : A \to \mathbb{R}$ define a basis change by a unimodular $n \times n$ matrix G by $G^*u : G^{-1}A \to \mathbb{R}$ via $G^*u(\mathbf{y}) := u(G\mathbf{y})$. Then*

1. *A bundle is demanded under the original demand at a certain price iff an associated bundle is demanded under the transformed demand at an associated price; specifically: $\mathbf{x} \in D_u(\mathbf{p}) \iff G^{-1}\mathbf{x} \in D_{G^*u}(G^T\mathbf{p})$.*

[25] Note our definition does not consider the weights on facets. We could take these into account, by relaxing the condition that all vectors in $\mathcal{D}$ be primitive. Then, for every facet F (with weight ω_F and primitive integer normal $\mathbf{v}_F$), we could require either that $\omega_F\mathbf{v}_F \in \mathcal{D}$, or that $\omega_F\mathbf{v}_F = k\mathbf{v}$, for some $k \in \mathbb{Z}$ and some $\mathbf{v} \in \mathcal{D}$. The former approach would allow us to specify the precise weights that facets may possess; this may seem unnatural, since a higher weight facet can be considered as the limit of two lower weight facets as they come arbitrarily close together, and thus very similar agent demands would be classified differently. The latter approach would allow us to insist on certain non-concavities of demand, and is a more straightforward generalisation of our definition. However, since we are most interested in concave demands, we have not pursued either approach here.

[26] We will see later (Section 4.2) that this demand type is "strong substitutes" in the two-good case, which we will label $\mathcal{D}^2_{ss}$.

[27] A unimodular matrix G is an integer matrix with integer inverse; an action of G on bundles of goods corresponds to an action of G^T on prices.

2. *The TH of the transformed demand is a linear transformation of the original demand:* $\mathcal{T}_{G^*u} = \{G^T\mathbf{p} \mid \mathbf{p} \in \mathcal{T}_u\}$*;*

3. *The same linear transformation applies to demand types:* $u(\cdot)$ *is of (concave) demand type* $\mathcal{D}$ *iff* $G^*u(\cdot)$ *is of (concave) demand type* $G^T\mathcal{D} = \{G^T\mathbf{v} \mid \mathbf{v} \in \mathcal{D}\}$*.*

Proof. See Appendix B.1. □

4.2 Substitutes

We use the standard definitions of "(ordinary) substitutes" and "strong substitutes", as used in the seminal papers by Ausubel and Milgrom (2002), and Milgrom and Strulovici (2009), respectively.[28]

Definition 4.3. Let $A \subsetneq \mathbb{Z}^n$ be finite, and $u : A \to \mathbb{R}$ be a valuation:

1. goods are *ordinary substitutes* if for any prices $\mathbf{p}' \geq \mathbf{p}$ such that $\#D_u(\mathbf{p}) = \#D_u(\mathbf{p}') = 1$, if $\{\mathbf{x}\} = D_u(\mathbf{p})$ and $\{\mathbf{x}'\} = D_u(\mathbf{p}')$ then $x'_k \geq x_k$ for all k such that $p_k = p'_k$.[29]

2. goods are *strong substitutes* if, when we consider every unit of every good as a separate good, then they are ordinary substitutes.[30]

We define two corresponding demand types, $\mathcal{D}^n_{os}$ and $\mathcal{D}^n_{ss}$; we will show $\mathcal{D}^n_{os}$ and *concave* $\mathcal{D}^n_{ss}$ are exactly equivalent to "ordinary substitutes" and "strong substitutes", respectively.

Definition 4.4.

1. $\mathcal{D}^n_{os}$ consists of those primitive integer vectors in $\mathbb{Z}^n$ with at most one positive and at most one negative coordinate entry, and all others zero.

2. $\mathcal{D}^n_{ss}$ consists of those vectors in $\mathbb{Z}^n$ with at most one $+1$ and at most one -1 coordinate entry, and all others zero.

[28] That is, we call "ordinary substitutes", precisely what Ausubel and Milgrom (2002) simply call "substitutes". We hope this increases clarity (since others loosely refer to substitutes in other ways). Note, in particular, that Ausubel and Milgrom's definition is *not* identical to that of Kelso and Crawford (1982) when there are multiple units of three or more goods. (See Baldwin and Klemperer, 2012; the definitions *are* equivalent in the simpler cases $A = \{0,1\}^n$–see Hatfield et al., 2011–and $n = 2$.) Milgrom and Strulovici (2009) call Kelso and Crawford's original definition "weak substitutes", but this is in fact a *stronger* definition of substitutes than Ausubel and Milgrom's. The latter definition (that we follow) seems most natural in the general case, and is also equivalent to several properties that seem to naturally characterise "standard" substitutes, and to the indirect utility function ($\max_{\mathbf{x}\in A}\{u(\mathbf{x})-\mathbf{p}.\mathbf{x}\}$) being submodular. We discuss these issues in detail in Baldwin and Klemperer (2012).

[29] See Ausubel and Milgrom (2002).

[30] See Milgrom and Strulovici (2009). That this definition is equivalent to theirs follows from Hatfield et al. (2011, Theorem A.1).

Any change in demands from one UDR to another can be divided up into a series of crossings of individual facets, at each of which demand changes in a direction prescribed by the demand type. So it is straightforward that a valuation of demand type $\mathcal{D}^n_{os}$ is an ordinary substitutes valuation, and vice versa. We use results in Milgrom and Strulovici (2009) and Hatfield et al. (2011 Theorem A.1) to show *concave* demand type $\mathcal{D}^n_{ss}$ is equivalent to strong substitutes:

Proposition 4.5.

1. *A valuation is of demand type $\mathcal{D}^n_{os}$ iff it is an ordinary substitutes valuation.*

2. *A valuation is of concave demand type $\mathcal{D}^n_{ss}$ iff it is a strong substitutes valuation.*

Proof. See Appendix B.1. □

So we can straightforwardly identify whether goods are ordinary or strong substitutes from their 'demand type'.

It is immediate, for example, that the examples of Figures 1, 4a, 12a, 12b, and 13a are all of type $\mathcal{D}^2_{ss}$, while our 3-dimensional example, Figure 5, has demand type $\mathcal{D}^3_{ss}$. However, Example 2.9 (Figure 3) has a facet with normal $(-1, 2)$ (the line segment between the prices (4,3) and (6,4)), in addition to facets with normals $(1, 0), (0, 1)$, and $(1, -1)$, and so is not of type $\mathcal{D}^2_{ss}$, but *is* of type $\mathcal{D}^2_{os}$, as is the example of Figure 13b.

4.3 Complements

"Complements" can be defined analogously to the Definition 4.3.1 of "ordinary substitutes":

Definition 4.6. Let $A \subsetneq \mathbb{Z}^n$ be finite, and let $u : A \to \mathbb{R}$ be a valuation. Goods are *complements* if, for any prices $\mathbf{p}' \geq \mathbf{p}$ such that $\#D_u(\mathbf{p}) = \#D_u(\mathbf{p}') = 1$, if $\{\mathbf{x}\} = D_u(\mathbf{p})$ and $\{\mathbf{x}'\} = D_u(\mathbf{p}')$ then $x'_k \leq x_k$ for all k such that $p_k = p'_k$.

Similarly to Definition 4.4.1 we define a corresponding demand type:

Definition 4.7. $\mathcal{D}^n_c$ consists of those primitive integer vectors in $\mathbb{Z}^n$ whose non-zero coordinate entries are all of the same sign.

As in Proposition 4.5.1, the facet normal at every crossing of facets that is part of the change in demands from one UDR to another prescribes the complementary property, so it is elementary that:

Proposition 4.8. *A valuation is of demand type $\mathcal{D}^n_c$ iff it is a complements valuation.*

Proof. See Appendix B.1. □

The examples of Figures 4b, 12b, 12c, 13c and 13d are all of type $\mathcal{D}^2_c$.

Note that although complements are often thought of as directly analogous to (ordinary) substitutes–as they are in two dimensions–this is not true if there are more than two goods. The case of complements permits facet normals with *any* number of non-zero entries, whereas substitutes permits at most two non-zero entries.

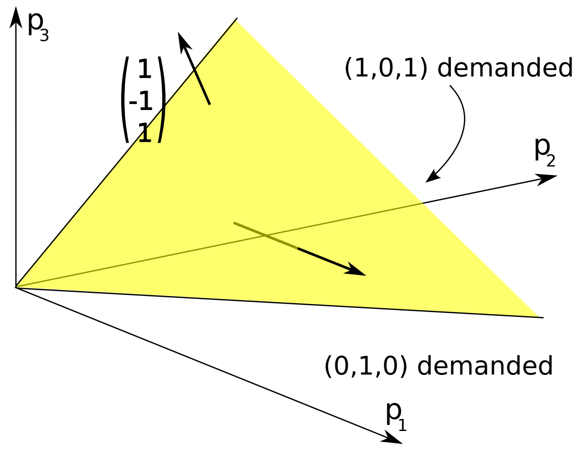


Figure 14: A facet with normal $(1, -1, 1)$: increasing either p_1 (as shown with an arrow) or p_3 demonstrates complementarities between goods 1 and 3.

The reason is that with substitutes, if any one good could trade-off against two others at the same price, it would necessarily follow that the two other goods were complementary. Even when all goods are mutual substitutes, there can never be trade-offs between more than two of them across a single facet: if more than two facet normal coordinate entries are non-zero, then at least two must have the same sign, so there are complementarities between the corresponding goods.

Consider, for example, Figure 14, in which there is a facet with normal (1,-1,1), defined by $\{\mathbf{p} \in \mathbb{R}^3 \mid p_1 + p_3 = p_2; p_1,p_2,p_3 \geq 0\}$: an increase in the price of *either* good 1 or good 3 that moves from the UDR with $p_1 + p_3 < p_2$ to the UDR with $p_1 + p_3 > p_2$ reduces demand for *both* goods. So, despite the symmetry between Definitions 4.3.1 and 4.6, complements allows far more degrees of freedom than does substitutes. One benefit of our way of classifying demand "types" is that it makes this lack of symmetry between substitutes and complements very clear.

4.4 Additively Separable Demand

Additively separable demand corresponds to an extremely simple demand type:

Definition 4.9. $\mathcal{D}_a^n$ consists of the coordinate vectors $\{\mathbf{e}^i \mid i = 1, \ldots, n\}$ in $\mathbb{Z}^n$.

In the additively separable case, a change in the price of one good will never affect demand for any other good. So it is not hard to show:

Proposition 4.10. *A valuation is of concave demand type $\mathcal{D}_a^n$ iff it is additively separable.*

Proof. See Appendix B.1. □

Note that being additively separable is a more stringent condition than being both (ordinary) substitutes and complements: neither of the latter conditions require a valuation to be concave, but an additively separable valuation necessarily is. A simple example of a valuation of type $\mathcal{D}_a^2$ which is not concave (or additively separable) is: $A = \{0, 1, 2\}^2$, and

$$u(x, y) = \begin{cases} x + y & (x, y) \in A, (x, y) \neq (1, 1) \\ 0 & (x, y) = (1, 1). \end{cases}.$$

4.5 Generalised Gross Substitutes and Complements (cf., Sun and Yang, 2006)

We define "generalised gross substitutes and complements" (GGSC) for the extension of Sun and Yang's (2006, see also 2009) definition of "gross substitutes and complements" to permit multiple units of goods. First recall:

Definition 4.11 (Sun and Yang, 2006, Definition 2.1)**.** A valuation $u : \{0, 1\}^{n_1+n_2} \to \mathbb{R}$ is a *gross substitutes and complements valuation (in the sense of Sun and Yang)* if, for any price $\mathbf{p}$ and any $\mathbf{p}' = \mathbf{p} + \delta \mathbf{e}^i$ where $\delta > 0$, and any $\mathbf{x} \in D_u(\mathbf{p})$: if $i \leq n_1$ then there exists $\mathbf{x}' \in D_u(\mathbf{p}')$ such that $x'_k \geq x_k$ for all $k \leq n_1$ such that $k \neq i$, and $x'_k \leq x_k$ for all $k > n_1$; and if $i > n_1$ then there exists $\mathbf{x}'' \in D_u(\mathbf{p}'')$ such that $x''_k \leq x_k$ for all $k \leq n_1$, and $x''_k \geq x_k$ for all $k > n_1$ such that $k \neq i$.

We will write I_{n_1,n_2} for the $(n_1+n_2) \times (n_1+n_2)$ matrix $I_{n_1,n_2} := \begin{pmatrix} -I_{n_1} & \mathbf{0} \\ \mathbf{0} & I_{n_2} \end{pmatrix}$ where I_{n_i} is the $n_i \times n_i$ identity matrix, $i = 1, 2$. Recall from Proposition 4.2 that, if $A \subsetneq \mathbb{Z}^{n_1+n_2}$ then, for any $u : A \to \mathbb{R}$, we define the valuation $I^*_{n_1,n_2} u : I^{-1}_{n_1,n_2} A \to \mathbb{R}$ via $I^*_{n_1,n_2} u(\mathbf{y}) = u(I_{n_1,n_2} \mathbf{y})$ for all $\mathbf{y} \in I^{-1}_{n_1,n_2} A$. Now we define:

Definition 4.12. Let $A \subsetneq \mathbb{Z}^n$ be finite, and let $u : A \to \mathbb{R}$ be a valuation. Goods are *generalised gross substitutes and complements (GGSC)* if the goods may be reordered such that, for some $n_1 + n_2 = n$, the valuation $I^*_{n_1,n_2} u$ is a strong substitute valuation.

The corresponding demand type we define is as follows:

Definition 4.13. $\mathcal{D}^{n_1,n_2}_{GGSC}$ is the following set of vectors in $\mathbb{Z}^{n_1+n_2}$

$$\{\mathbf{e}^i, \mathbf{e}^j, \mathbf{e}^i - \mathbf{e}^{i'}, \mathbf{e}^i + \mathbf{e}^j, \mathbf{e}^j - \mathbf{e}^{j'} \mid i, i' \in \{1, \ldots, n_1\}, j, j' \in \{n_1 + 1, \ldots, n_1 + n_2\}\}.$$

It follows straightforwardly that:

Proposition 4.14. *A valuation is a GGSC valuation iff the goods may be reordered such that, for some $n_1 + n_2 = n$, it is of concave type $\mathcal{D}^{n_1,n_2}_{GGSC}$.*

Proof. It is not hard to see that $\mathcal{D}^{n_1,n_2}_{GGSC} = I^T_{n_1,n_2} \mathcal{D}^n_{ss}$. So the result follows by Definition 4.12 and by Proposition 4.2. □

Sun and Yang (2006, Section 3) have shown definitions 4.11 and 4.12 coincide when $A = \{0, 1\}^n$, so this interpretation applies also to their demand structure.

5 Aggregate Demand and Equilibrium

We now consider aggregate demand across many agents. In particular, we precisely identify the demand types for which competitive equilibrium always exists.

5.1 The structure of aggregate demand

We now have a finite set J of agents. Each agent j has a valuation u^j of integer bundles in a finite set A_j, so the bundles of interest on aggregate are $A := \{\sum_{j\in J} \mathbf{x}^j \mid \mathbf{x}^j \in A_j\}$, which we shall refer to as the *support* of the aggregate valuation.[31] The aggregate demand at any price $\mathbf{p}$ is simply

$$D_{\{u^j\}}(\mathbf{p}) := \left\{ \sum_{j\in J} \mathbf{x}^j \mid \mathbf{x}^j \in D_{u^j}(\mathbf{p}) \right\}. \tag{4}$$

One way to find aggregate demand is to start with the valuation functions $u^j(\cdot)$, combine them to give an 'aggregate valuation function', and then proceed in exactly the same way as for individual demand. It is standard (see Appendix B.2) that if agents' preferences are quasilinear then one attains an aggregate valuation function $U : A \to \mathbb{R}$ as the greatest sum of valuations that can be attained by dividing any bundle $\mathbf{y} \in A$ between the agents:

$$U(\mathbf{y}) := \max \left\{ \sum_{j\in J} u^j(\mathbf{x}^j) \mid \mathbf{x}^j \in A_j, \sum_{j\in J} \mathbf{x}^j = \mathbf{y} \right\},$$

and:

Proposition 5.1. *$D_{\{u^j\}}(\mathbf{p}) = D_U(\mathbf{p})$ for all $\mathbf{p} \in \mathbb{R}^n$.*

So we henceforth refer to $D_{\{u^j\}}(\mathbf{p})$ using the simpler notation $D_U(\mathbf{p})$.

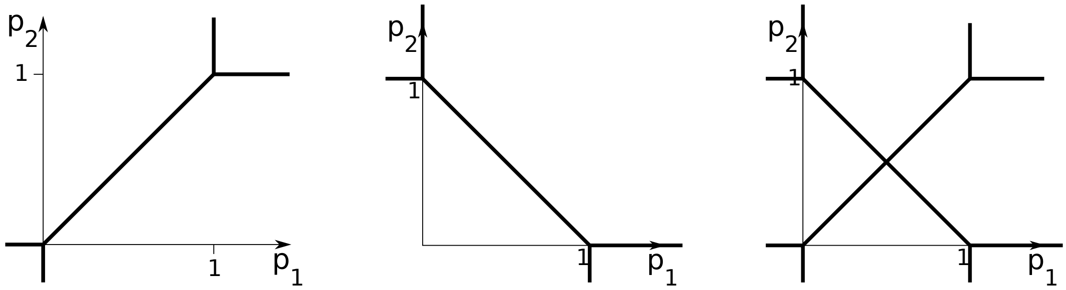
However, the problem with this approach is that $U(\cdot)$ is very hard to work with–to find any value of $U(\mathbf{y})$, we need to consider all possible partitions of y among the agents, which is both time-consuming and unintuitive.

It is straightforward, on the other hand, to start with the individual THs, $\mathcal{T}_{u^j}$, combine them to form an aggregate TH, $\mathcal{T}_{\{u^j\}}$, and find information about aggregate demand from that. Recall that the underlying set of $\mathcal{T}_{u^j}$ is those prices at which demand u^j is non-unique. So, since aggregate demand $D_U(\mathbf{p})$ is unique iff all individual demands $D_{u^i}(\mathbf{p})$ are, the underlying set of $\mathcal{T}_{\{u^j\}}$ is just the union of all the $\mathcal{T}_{u^j}$. Figure 15 illustrates this for the aggregate of the two agents' demands in our simple substitutes and complements example, Example 2.10.

$\mathcal{T}_{\{u^j\}}$ inherits the structure of a proper rational polyhedral complex from the individual THs, although the cells will not in general be exactly the same: if cell interiors from two different agents intersect, the cells are split up into new, smaller cells in $\mathcal{T}_{\{u^j\}}$ with a new, lower-dimensional, cell at their intersection. For example, in Figure 15b, the point $(\frac{1}{2}, \frac{1}{2})$ is a 0-cell, on the boundary of four *distinct* 1-cells.

It is easy to see that $\mathcal{T}_{\{u^j\}}$ also inherits a balanced weighting from the weightings of the individual THs. For any facet F of $\mathcal{T}_{\{u^j\}}$, let its weighting $w_{\{u^j\}}(F)$ be $\sum_{j\in J} w_j(F)$, in which $w_j(F)$ is the weight of the facet $F_j \supseteq F$ of $\mathcal{T}_{u^j}$, or $w_j(F) = 0$ if no facet $F_j \supseteq F$ of $\mathcal{T}_{u^j}$ exists. Since each individual TH is balanced, adding weightings in this

[31] We could alternatively consider each agent as having a valuation over the full support of the aggregate valuation A by letting $u^j(\mathbf{x}) := \max\{u(\mathbf{y}) \mid \mathbf{y} \in A_j,\, y_i \leq x_i,\, i = 1, \ldots, n\}$ for any $\mathbf{x} \in A$ for which this set is non-empty, and $u^j(\mathbf{x}) = -\infty$ otherwise.



(a) $\mathcal{T}_{u^1}$ (b) $\mathcal{T}_{u^2}$ (c) $\mathcal{T}_{\{u^j\}}$

Figure 15: (a) and (b) the THs of the individual demands of Example 2.10; (c) the TH of the aggregate of the two demands of Example 2.10.

way creates a balanced weighting.[32] And the change in aggregate demand as we cross a facet is just the sum of changes in individual demand.

So, since the underlying sets of $\mathcal{T}_{\{u^j\}}$ and $\mathcal{T}_U$ are the same, and so are their weightings, it follows (see Appendix B.2) that *as THs*,

Proposition 5.2. $\mathcal{T}_{\{u^j\}} = \mathcal{T}_U$.

So we will henceforth also refer to the aggregate TH, $\mathcal{T}_{\{u^j\}}$, using the simpler notation $\mathcal{T}_U$.

Thus simply "adding" the individual THs yields the aggregate TH. If we know what is demanded in one UDR then, as before, we immediately know what is demanded in all the UDRs, without needing to directly consider the function U. And it is immediate that demand 'type' is preserved under aggregation:

Corollary 5.3. *Valuations u^j are of demand type $\mathcal{D}$ for all $j \in J$ iff the aggregate demand $\mathcal{T}_U$ is of demand type $\mathcal{D}$.*

Proof. This is immediate from Proposition 5.2 and the definition of $\mathcal{T}_{\{u^j\}}$. □

However, it is *not* the case that concavity of each individual demand implies concavity of the aggregate demand. (We will exhibit a simple example of this failure in Example 5.11.) And we have seen (Lemma 2.5) that if the function U is not concave, then there exists a bundle in A that is never demanded.

Furthermore, we cannot generally infer from an aggregate TH whether there is a bundle that is never the aggregate demand–recall that the geometric construction does not tell us the precise demand set, $D_U(\mathbf{p})$, at all prices $\mathbf{p} \in \mathcal{T}_{\{u^j\}}$, so it is ambiguous from the geometry whether any integer vectors in $\operatorname{Conv} D_U(\mathbf{p})$ that are not vertices of $\operatorname{Conv} D_U(\mathbf{p})$ are in $D_U(\mathbf{p})$ (see Corollary 3.5).

[32] In more detail: let G be a $(n-2)$-cell in $\mathcal{T}_{\{u^j\}}$, let $F_1, \ldots, F_l$ be the facets adjacent to G, and let $\mathbf{v}_{F_k}$ be primitive integer vectors for each, chosen according to a coherent orientation. Then for every agent j, the equation $\sum_{k=1}^{l} w_j(F_k)\mathbf{v}_{F_k} = 0$ holds: if G is contained in an $(n-2)$ -cell of $\mathcal{T}_{u^j}$ then, this follows from $\mathcal{T}_{u^j}$ being balanced; if G is contained only in a single facet of $\mathcal{T}_{u^j}$ then the only non-zero terms in this sum are those which first add and then subtract the weight of this facet to j; if $G \cap \mathcal{T}_{u^j} = \emptyset$ then the expression is identically zero. We conclude $\sum_{k=1}^{l} w_{\{u^j\}}(F_k)\mathbf{v}_{F_k} = \sum_{j \in J}\sum_{k=1}^{l} w_j(F_k)\mathbf{v}_{F_k} = 0$. Alternatively, one can see this by appealing to Appendix B.2, which confirms that the weightings are the same as those on $\mathcal{T}_U$ – being, of course, automatically balanced since it is the TH corresponding to $U(\cdot)$.

Of course, if there is a bundle which is not the aggregate demand of the agents for any price, then a competitive equilibrium does not exist when this is the bundle of goods available in the economy.

The remainder of this section therefore provides conditions which guarantee that a competitive equilibrium *always* exists, by providing conditions which guarantee that the aggregate valuation $U(\cdot)$ is concave (without needing to explicitly calculate $U(\cdot)$). In particular, we are interested in the existence of equilibrium for agents with specified demand types, as defined in Section 4:

Definition 5.4. A (concave) demand type $\mathcal{D}$ *always has a competitive equilibrium* if, for every set of agents with (concave) demands of type $\mathcal{D}$, and for an economy endowed with any bundle in the support of the aggregate valuation, a competitive equilibrium exists.

Note that there always exist *some* collections of agents with demands of type $\mathcal{D}$ which *do* have a competitive equilibrium for any supply in their support, whether or not the type $\mathcal{D}$ 'always has a competitive equilibrium'.[33] Note also that since the demand of a single agent with non-concave valuation function fails to always have a competitive equilibrium, we are only interested in concave demand types here.

A benefit of our method of categorising demand types is that it is straightforward that:

Proposition 5.5. *Always having a competitive equilibrium is a property that is preserved under unimodular basis changes.*

Proof. See Appendix B.2 □

5.2 When does Competitive Equilibrium exist?

This section proves a theorem, inspired by "intersection multiplicities" in tropical geometry, which identifies precisely which demand types always have a competitive equilibrium. Our assumptions about agents' preferences are weaker than in the existing literature, so our "necessary and sufficient" condition for equilibrium is correspondingly more general. In particular (see Section 5.3.3) it is *not* necessary for all agents to have strong substitute demands (or some basis change thereof) for equilibrium to always exist.[34]

Throughout, we write "the *determinant* of vectors $\mathbf{w}^1, \ldots, \mathbf{w}^n$" to mean the determinant of the $n \times n$ matrix which has these vectors as its columns.[35] And we say that a linearly independent set $\{\mathbf{w}^1, \ldots, \mathbf{w}^s\}$ of vectors is "an *integer basis* for the subset they

[33] Trivially, a set of identical agents with concave demands of the type $\mathcal{D}$ *does* always have a competitive equilibrium.

[34] For example, results such as those of Kelso and Crawford (1982), Hatfield and Kojima (2008), and Hatfield et al. (2012) are necessary 'in the maximal domain sense', in Hatfield et al. (2012)'s words. That is, in our language, they show that equilibrium always exists for some demand type $\mathcal{D}$, but that if one agent has preferences outside of $\mathcal{D}$ then this may fail.

[35] Changing the order of the vectors may change the sign of the determinant, so strictly speaking the determinant is a property of an ordered n-tuple of vectors. This detail does not concern us as we are only ever interested in the absolute values of determinants.

span" if, whenever $\mathbf{y} \in \mathbb{Z}^n$ can be written as $\sum_{i=1}^{s} a_i \mathbf{w}^i$, in which $a_i \in \mathbb{R}$, then in fact $a_i \in \mathbb{Z}$ for $i = 1, \ldots, s$.

Theorem 5.6. *A concave demand type $\mathcal{D}$ always has a competitive equilibrium iff every linearly independent set of vectors from $\mathcal{D}$ are an integer basis for the subspace they span.*

It is a standard result that our condition on $\mathcal{D}$ is equivalent to the condition that any s-dimensional parallelepiped, whose edges are s linearly independent vectors in $\mathcal{D}$, contains no integer point (either in its boundary or in its interior) aside from its vertices–this equivalence will be helpful in understanding our result. But this condition holds iff the s-dimensional *volume* of the parallelepiped is 1 and, when $s = n$, this volume is simply the (absolute value of the) determinant of the vectors along its edges. So if the set of aggregate demands is in the same dimension as the number of goods, we can re-state the theorem in a form that is easier to check:

Corollary 5.7. *With n goods, a concave demand type $\mathcal{D} = \{\mathbf{v}^1, \ldots, \mathbf{v}^r\}$, in which $\mathbf{v}^1, \ldots, \mathbf{v}^r$ span $\mathbb{R}^n$, always has a competitive equilibrium iff every subset of n vectors from $\mathcal{D}$ has determinant 0 or ± 1.*

In the more general case of Theorem 5.6 we allow demand types $\mathcal{D}$ that ignore some directions of good availability. In such a $\mathcal{D}$ there are no collections of n linearly independent vectors, so every subset of n vectors has determinant 0, and the check of Corollary 5.7 tells us nothing. In this case, however, we can use one of the equivalent conditions in Remark 5.8.2 and .3:

Remark 5.8. The following are equivalent, for a set of s linearly independent vectors in $\mathbb{Z}^n$:

1. they are an integer basis for the subspace they span;
2. they can be extended to a basis for $\mathbb{R}^n$, of integer vectors, with determinant ± 1;
3. among the determinants of all the $s \times s$ matrices consisting of s rows of the $n \times s$ matrix whose columns are these s vectors, the greatest common factor is 1.

Proofs of these facts may be found in Cassels (1959).[36]

We first show Theorem 5.6's condition is necessary by presenting a class of examples: whenever a set of s linearly independent vectors fails the condition, an example from this class exhibits failure of competitive equilibrium. Take such a set of s vectors, and fix a price $\mathbf{p}$. Suppose we have s distinct corresponding agents such that, at price $\mathbf{p}$, each agent is indifferent between precisely two bundles, and those bundles differ by the corresponding vector from the set. Thus each individual TH has a facet with $\mathbf{p}$ in its interior, and with the corresponding vector as its normal vector. Suppose also that the individual TH of no agent other than these s agents passes through $\mathbf{p}$.[37] Then there

[36] 1 $\Leftrightarrow$ 2 follows from Cassels (1959) Lemma I.1 and Corollary I.3. 1 $\Leftrightarrow$ 3 is Cassels (1959) Lemma I.2.

[37] Our 'determinant condition' is equivalent to the *tropical intersection multiplicity* being greater than one in such a case (see e.g. Osserman and Payne, 2010).

always exists a possible bundle which is not demanded at any price. In fact, the SNP face Conv $D_U(\mathbf{p})$ is a parallelepiped whose edges are the s vectors in question; it *does* contains an integer point which is not at a vertex, and this integer point is precisely the bundle which is never demanded. In detail:

Proposition 5.9. *Consider $s \leq n$ agents each of whose demand set includes precisely 2 bundles at price $\mathbf{p}$, i.e., $\#D_{u^i}(\mathbf{p}) = 2$, for $i = 1, \ldots, s$. Write $\mathbf{v}^i$ for the difference between the two bundles demanded by agent i (so $\mathbf{v}^i$ is normal to i's facet of demand at $\mathbf{p}$). Write U for the aggregate valuation. Suppose the s vectors $\mathbf{v}^1, \ldots, \mathbf{v}^s$ are linearly independent but not an integer basis for the subspace they span. Then there exists an integer bundle in* Conv $D_U(\mathbf{p})$ *which is not demanded at any price.*

Proof. Each agent i's demand at $\mathbf{p}$ has the form $D_{u^i}(\mathbf{p}) = \{\mathbf{y}^i + \delta_i \mathbf{v}^i \mid \delta_i \in \{0, 1\}\}$, where $\mathbf{y}^i$ is the bundle demanded on the appropriate side of the TH facet. So the set of bundles demanded on aggregate at $\mathbf{p}$ is

$$D_U(\mathbf{p}) = \left\{\mathbf{y} + \delta_1 \mathbf{v}^1 + \cdots + \delta_s \mathbf{v}^s \mid \delta_i \in \{0, 1\}; i = 1, \ldots, s\right\},$$

where $\mathbf{y} = \sum_i \mathbf{y}^i$. That is, $D_U(\mathbf{p})$ is precisely the vertices of an s-dimensional parallelepiped in $\mathbb{Z}^n$ (since the $\mathbf{v}^i$ are linearly independent) and, in particular, no bundle not at the vertex of the parallelepiped is in $D_U(\mathbf{p})$. But it follows from the assumptions that this parallelepiped contains a lattice point (that is, an integer bundle) not at one of its vertices.[38] For we know there exists a vector $\mathbf{v} \in \mathbb{Z}^n$ such that $\mathbf{v} = \sum_i \beta_i \mathbf{v}^i$ with β_i not all in $\mathbb{Z}$. Subtracting, for each i, the integer part of β_i times $\mathbf{v}^i$ yields a vector $\tilde{\mathbf{v}} \in \mathbb{Z}^n$ such that $\mathbf{y} + \tilde{\mathbf{v}}$ is within the parallelepiped described. That is, there exists an integer bundle $\in$ Conv $D_U(\mathbf{p})$ but $\notin D_U(\mathbf{p})$, and by Lemma 3.1, such a bundle cannot be demanded at any price. $\square$

Thus Theorem 5.6's condition is necessary; we now turn to the question of its sufficiency.

A demand type, $\mathcal{D}$, always has a competitive equilibrium iff all integer bundles (i.e., all lattice points) in the SNP of aggregate demand are demanded for some price. It is immediate that any integer bundle that is at a vertex of the SNP is demanded. Furthermore, any integer lattice point in the SNP of aggregate demand that is not a vertex is "hidden" inside the corresponding intersection of the individual agents' THs. So the question is whether all the integer bundles that are in the convex hull of the demands at an intersection of THs of agents with demand of type $\mathcal{D}$ are always demanded.

We show the "if" part of the Theorem in two stages: first, we show in Proposition 5.10 that all the integer bundles in the convex hull of the demands at any "nice" intersection of agents' THs are always demanded.

We will make our definition of a "nice" intersection precise in the statement of Proposition 5.10 below. We will see that it covers any generic intersection at a single price. For example, in two dimensions, two lines crossing at a single point is "nice", but having two coincident lines is not "nice", and nor is three lines crossing at a single point (which is non-generic); in three dimensions, either three planes meeting in a single point, or a line meeting a plane in a single point is "nice". The important property of

[38]This result amounts to a simple case of Minkowski's theorem, for which see e.g. Cassels (1959) Chapter III.

"nice" intersections is that the changes in bundles considered by the different agents (as each agent crosses between different regions of its TH) are always linearly independent. This means that, if we consider any change in the aggregate demand at this price point, we can straightforwardly and uniquely apportion it between the individual agents by simply giving each individual agent that part of the aggregate change that follows its direction of change.

Our proof of Proposition 5.10 observes that this fact implies that at any "nice" intersection price, any integer bundle in the convex hull of aggregate demand can also be uniquely partitioned into separate components of the bundle that are demanded by the different individual agents; it then follows that each of these separate components is in the convex hull of the corresponding individual agent's demand at the given price. Furthermore, the condition of Theorem 5.6 tells us precisely that any integer change in aggregate demand at the price must correspond to a (possibly zero) integer change in each agent's allocation of each good. The concavity of each individual agent's valuation then means that each separate component of the total bundle is demanded by an individual agent, and therefore that the aggregate bundle is also demanded. So we now prove this in Proposition 5.10, for an appropriate definition of "nice".

Proposition 5.10. *Suppose price* $\mathbf{p}$ *is in the interior of an* $(n-k_i)$*-cell* C_i *of the TH* $\mathcal{T}_{u^i}$ *of each of* s *agents* $i = 1, \ldots, s$*, who have concave valuations* u^i*, and together have aggregate valuation* $\tilde{U}$*. Then every integer bundle in* $\operatorname{Conv} D_{\tilde{U}}(\mathbf{p})$ *is demanded at* $\mathbf{p}$ *if each* C_i *is a subset of the intersection of a set of facets* $F^i_1, \ldots, F^i_{k_i}$ *of* $\mathcal{T}_{u^i}$ *(not necessarily comprising all facets of* $\mathcal{T}_{u^i}$ *that pass through* C_i*) with primitive integer normal vectors* $\mathbf{v}^i_1, \ldots, \mathbf{v}^i_{k_i}$ *and* $\{\mathbf{v}^i_j \mid i = 1, \ldots, s; j = 1, \ldots, k_i\}$ *are an integer basis for the subspace of* $\mathbb{R}^n$ *they span.*

Proof. Agent i demands at $\mathbf{p}$ precisely the bundles demanded throughout the $(n-k_i)$-cell C_i, which corresponds to a k_i-dimensional polytope Δ_i in the SNP of agent i. Moreover, Δ_i possesses an edge in direction $\mathbf{v}^i_j$ for $j = 1, \ldots, k_i$; each corresponds to the facet F^i_j. Thus, if $\mathbf{y}^i$ is some integer bundle in $D_{u^i}(\mathbf{p})$, then (by a dimension count) the affine span of Δ_i is precisely $\left\{\mathbf{y}^i + \sum_{j=1}^{k_i} \beta^i_j \mathbf{v}^i_j \mid \beta^i_j \in \mathbb{R} \text{ for } j = 1, \ldots, k_i\right\}$, and in particular, $D_{u^i}(\mathbf{p})$ is contained in this set.

Thus, using equation (4) we may express aggregate demand among these agents as $D_{\tilde{U}}(\mathbf{p}) = \left\{\mathbf{y} + \sum_{i=1}^{s} \sum_{j=1}^{k_i} a^i_j \mathbf{v}^i_j \mid \mathbf{y}^i + \sum_{j=1}^{k_i} a^i_j \mathbf{v}^i_j \in D_{u^i}(\mathbf{p}) \text{ for } i = 1, \ldots, s\right\}$, where $\mathbf{y} := \sum_{i=1}^{s} \mathbf{y}^i$.

Now, suppose $\mathbf{x}$ is an integer bundle in $\operatorname{Conv} D_{\tilde{U}}(\mathbf{p})$. Then $\mathbf{x} - \mathbf{y}$ is in the span of the $\mathbf{v}^i_j$. But since they are an integer basis for their span, we can write $\mathbf{x} - \mathbf{y} = \sum_{i=1}^{s} \sum_{j=1}^{k_i} b^i_j \mathbf{v}^i_j$, for some $b^i_j \in \mathbb{Z}$. So we can define $\mathbf{x}^i := \mathbf{y}^i + \sum_{j=1}^{k_i} b^i_j \mathbf{v}^i_j$, and know that $\mathbf{x}^i \in \mathbb{Z}^n$.

But we also know $\mathbf{x}^i \in \operatorname{Conv} D_{u^i}(\mathbf{p})$. To see this, observe that since $\mathbf{x} \in \operatorname{Conv} D_{\tilde{U}}(\mathbf{p})$, we can write $\mathbf{x} - \mathbf{y} = \sum_{\beta} \sum_{i=1}^{s} \sum_{j=1}^{k_i} \lambda_\beta a^i_{j,\beta} \mathbf{v}^i_j$ for some finite set of weights $\lambda_\beta \in [0,1]$ such that $\sum_\beta \lambda_\beta = 1$ and such that $\mathbf{y}^i + \sum_{j=1}^{k_i} a^i_{j,\beta} \mathbf{v}^i_j \in D_{u^i}(\mathbf{p})$ for each agent i and for each β. But since the $\mathbf{v}^i_j$ are linearly independent, there is an unique way to write $\mathbf{x} - \mathbf{y}$ as a weighted sum of the $\mathbf{v}^i_j$, so $b^i_j = \sum_\beta \lambda_\beta a^i_{j,\beta}$, and so $\mathbf{x}^i = \mathbf{y}^i + \sum_{j=1}^{k_i} b^i_j \mathbf{v}^i_j = \mathbf{y}^i + \sum_{j=1}^{k_i} \sum_\beta \lambda_\beta a^i_{j,\beta} \mathbf{v}^i_j \in \operatorname{Conv} D_{u^i}(\mathbf{p})$.

So $\mathbf{x}^i$ is an integer vector in $\operatorname{Conv} D_{u^i}(\mathbf{p})$. By concavity of u^i there exists *some* price at which $\mathbf{x}^i$ is demanded by agent i (Lemma 2.5), and so by Lemma 3.1 we know $\mathbf{x}^i \in D_{u^i}(\mathbf{p})$. Thus $\mathbf{x} = \sum_{i=1}^{s} \mathbf{x}^i \in D_{\tilde{U}}(\mathbf{p})$. That is, $\mathbf{x}$ is demanded at $\mathbf{p}$, as required. □

Proposition 5.10 shows that Theorem 5.6's condition is sufficient if all the intersections of the TH are "nice". The second half of the proof of the "if" part of the Theorem shows that generically all TH intersections are "nice", and that any non-"nice" intersection is therefore close enough to being a "nice" intersection that Theorem 5.6's condition still suffices.

Consider an integer bundle that is "hidden" in the convex hull of aggregate demand at a price point in a *not*-nice intersection. If it is not demanded at this price, agents' aggregate utility from this bundle, at this price vector, must be strictly lower than their aggregate utility from any bundle that is demanded at this price. Since this bundle is a convex combination of other bundles that are demanded at this price vector, the aggregate valuation from the bundle in question is strictly lower than the same convex combination of the aggregate valuations of these other bundles. Let this aggregate valuation difference be ϵ.

Now consider perturbing all agents' valuation functions by arbitrarily small amounts, so that their TH undergoes a small translation in price space. It is straightforward, although somewhat tedious, to show that generically all the TH intersections are now "nice". So we can choose these small perturbations so this holds; additionally, we ensure that no agent's valuation of any available bundle is affected by more than $\frac{\epsilon}{3m}$, in which m is the number of agents present.

If the condition of Theorem 5.6 is satisfied, the bundle in question is (by Proposition 5.10) demanded by agents with the perturbed valuation functions at some price. But the perturbation of the valuation functions cannot change the aggregate valuation from either this bundle, or the same convex combination of the aggregate valuation of the other bundles, by more than $\epsilon/3$. So the aggregate valuation from this bundle is still below the same convex combination of the aggregate valuation of the other bundles, and therefore the aggregate utility of this bundle is also still below the same convex combination of the aggregate utility of the other bundles at any prices (since at any prices, the cost of this bundle equals this convex combination of the cost of the other bundles). So we have a contradiction, and the lattice point must have been demanded at the original price point. That is, Theorem 5.6's condition is also sufficient for non-"nice" intersections.

We give the formal details of this part of the proof in Appendix B.2.

5.3 Examples

5.3.1 Examples of non-existence of equilibrium

We first illustrate our result with two simple examples of non-existence of equilibrium, whose demands fit the conditions of Proposition 5.9.

Example 5.11. [39] The simplest concave type of demand for which equilibrium need not exist has one agent for whom two goods are substitutes, and a second agent for whom

[39] Hatfield et al. (2012, Example 2) present essentially this example.

the same two goods are complements. So $\mathcal{D}$ can be represented by

$$D = \begin{pmatrix} 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$$

in which the first three column vectors together yield the substitutes demand, and the last three column vectors together yield complements demand. Trivially, the matrix formed by the first and last column has determinant 2, so equilibrium need not exist.

Our Example 2.10 is of this type: we repeat its valuation functions for the "substitutes agent" and "complements agent" respectively, below:

$x_1 = 1$	$x_1 = 0$	u^1
1	0	$x_2 = 0$
1	1	$x_2 = 1$

and

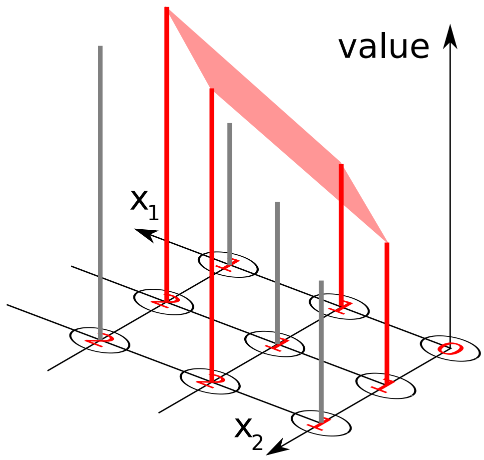
$x_1 = 1$	$x_1 = 0$	u^2
0	0	$x_2 = 0$
1	0	$x_2 = 1$

,

Note that *both* these valuation functions are concave. However, the aggregate valuation function, which we give in Figure 16a is *not* concave, as can be easily seen by observing that $(U(1,0)+U(0,1)+U(2,1)+U(1,2))/4 > U(1,1)$. This inequality is also apparent in Figure 16b which shows a 3-dimensional illustration of U together with the face of $\widehat{A}$ (see equation (3)) that corresponds to the price vector $(\frac{1}{2}, \frac{1}{2})$. It follows that all the

$x_1 = 2$	$x_1 = 1$	$x_1 = 0$	U
1	1	0	$x_2 = 0$
2	1	1	$x_2 = 1$
2	2	1	$x_2 = 2$

(a) Aggregate valuation.

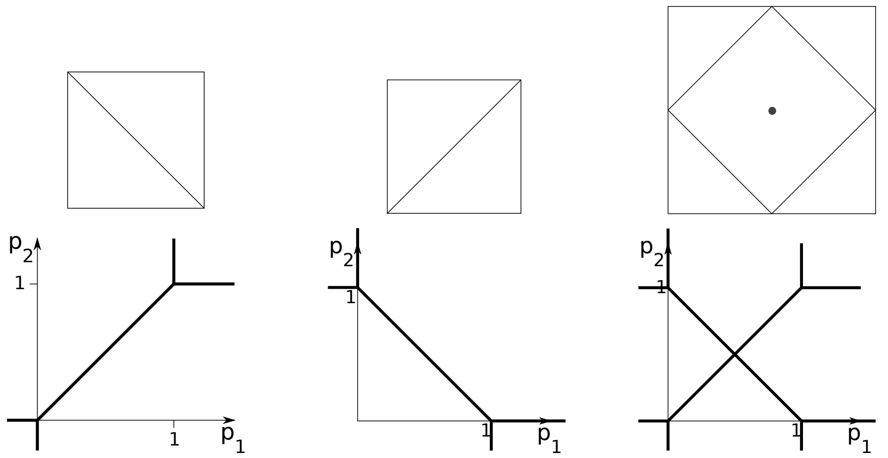


(b) 3 dimensional illustration of the aggregate valuation, showing the face of $\widehat{A}$ that corresponds to the price vector $(\frac{1}{2}, \frac{1}{2})$.

Figure 16: The aggregate valuation of Example 5.11.

bundles $(1,0)$, $(0,1)$, $(2,1)$, and $(1,2)$ are demanded at this price, while the bundle $(1,1)$ is "hidden" at the intersection of the diagonals of the TH at the price, $(\frac{1}{2}, \frac{1}{2})$, and is never demanded at any price. So aggregate demand is never $x_1 = x_2 = 1$. The SNP and the TH of the individual and aggregate demands are shown in Figure 17. Observe in Figure 17c that in the aggregate SNP the bundle $(1,1)$ is not a vertex, and the area of the diamond is $\det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2$.

Of course, our analysis only shows that equilibrium *may* not exist for this type of demand. Equilibrium would exist if, for example, the "complements" consumer had valuation 3 for the combination of 1 unit of each of x_1 and x_2. In that case the facets



(a) The "substitutes" SNP and TH.

(b) The "complements" SNP and TH.

(c) The aggregate SNP and TH.

Figure 17: The individual and aggregate SNPs and THs for Example 5.11.

corresponding to the vectors $(1,1)$ and $(1,-1)$ would not intersect, so Proposition 5.9 does not apply. We will return to this issue in Section 5.4.

Example 5.12. [40] Consider a set of "complements" consumers each of whom is only interested in a different pair of goods, and such that there is a cycle in the pairs of goods that these consumers wish for. That is, we can number both consumers and goods $1,\dots,n$, such that every consumer $i<n$ demands goods i and $i+1$, and consumer n demands goods n and 1. It is not hard to see that:[41]

$$\text{if } D = \begin{pmatrix} 1 & 0 & 0 & & & 0 & 1 \\ 1 & 1 & 0 & & & 0 & 0 \\ 0 & 1 & 1 & & & 0 & 0 \\ \cdot & 0 & 1 & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & & \cdot & 1 & 0 \\ 0 & 0 & 0 & & & 1 & 1 \end{pmatrix} \quad \text{then } \det D = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

So if n is odd, there exist agents with demands of this type such that equilibrium does not exist: such an example is, again, easy to construct following Proposition 5.9.

Indeed, one can see directly that equilibrium fails in the simplest symmetric case: if each consumer has valuation 1 for any allocation that includes the pair it desires, and valuation 0 for any other allocation, aggregate demand is never exactly 1 unit of each good. To see this, note that at least one good, w.l.o.g. good 1, would not be part of a pair. So $p_1 = 0$. Therefore $p_2 \geq 1$ (else consumer 1 would demand the pair of goods 1 and 2). So $p_2 = 1$, and therefore $p_3 = 0$, since otherwise good 2 would not be demanded, and consumer 2 therefore buys goods 2 and 3. Therefore $p_4 \geq 1$ (else consumer 3 would

[40] Sun and Yang (personal communication), and also Teytelboym (2012), have independently considered the demand described in this example, using alternative methods that extend Sun and Yang (2006), showing as we do that equilibrium always exists iff n is even. See also Footnote 42.

[41] To see this easily, expand by the first row: noting the "1"s in the first and the last column of that row, we have $\det D = 1(1) + (-1)^{n-1}(1)$.

demand goods 3 and 4). So $p_4 = 1$, and $p_5 = 0$, etc. In particular, $p_j = 0$ if j is odd. But in that case, consumer n wishes to buy goods n and 1, which is a contradiction.

On the other hand, if n is *even*, the columns of D are not linearly independent, but if we exclude the ith column, for any i, the remaining $n-1$ rows are linearly independent and can trivially be extended to n linearly independent vectors with determinant 1 by adding the column $\mathbf{e}^i$, so Theorem 5.6 then shows that equilibrium *always* exists. For example, in the simple symmetric case, $p_j = 0$ if j is odd and $p_j = 1$ if j is even, for all j, supports $q_i = 1$ for all i as an aggregate demand.

5.3.2 Strong substitutes and Generalised gross substitutes and complements

Recall from Section 4.2 that a valuation is 'strong substitutes', in the terminology introduced by Milgrom and Strulovici (2009), if every unit of every good is an ordinary substitute for every other unit of every good (including being an ordinary substitute for every other unit of the same good). We showed in Proposition 4.5.2 that strong substitutes are precisely are concave demand type $\mathcal{D}^n_{ss}$; the latter may be presented as $\{\mathbf{e}^i, \mathbf{e}^i - \mathbf{e}^j \mid i, j = 1, \ldots, n; i < j\}$ (see Section 4.2).

One of the pleasing properties that Milgrom and Strulovici (2009) showed for 'strong substitutes' is that equilibrium always exists:

Proposition 5.13 (Milgrom and Strulovici, 2009, Theorem 19). *Equilibrium always exists when agents' demands are strong substitutes.*

Our framework makes it particularly straightforward to confirm this result, by showing that $\mathcal{D}^n_{ss}$ satisfies the condition of Corollary 5.7:

Note first that any vector $\mathbf{v} = \mathbf{e}^i - \mathbf{e}^j$ satisfies $\mathbf{v}.\mathbf{1} = 0$, where $\mathbf{1} = (1, 1, \ldots, 1)^T$. So any set of n vectors that are all of the form $\mathbf{e}^i - \mathbf{e}^j$ does not have $\mathbf{1}$ in its span, so is not linearly independent and therefore has determinant 0. It follows that any set of linearly independent vectors in $\mathcal{D}^n_{ss}$ must include a coordinate vector $\mathbf{e}^i$ (or $-\mathbf{e}^i$). Now observe that the determinant of any matrix which has this set of vectors as its columns is non-zero (since the vectors are linearly independent), and also ± 1 times the $(n-1) \times (n-1)$ matrix formed when we delete row i and the column in which $\pm\mathbf{e}^i$ was placed. But since this $(n-1) \times (n-1)$ matrix therefore has non-zero determinant, its columns are linearly independent, and they are also vectors in $\mathcal{D}^{n-1}_{ss}$. So $\mathcal{D}^n_{ss}$ satisfies the determinant condition if $\mathcal{D}^{n-1}_{ss}$ does. But it is trivial that $\mathcal{D}^1_{ss}$ satisfies the condition so, by induction on n, $\mathcal{D}^n_{ss}$ satisfies the condition of Corollary 5.7 for all n.

Moreover, it is now trivial to reproduce:

Corollary 5.14 (Milgrom and Strulovici, 2009, Theorem 20). *If u^j is a strong substitute valuation for all $j \in J$, then the aggregate valuation U is a strong substitute valuation.*

Proof. If u^j is of concave demand type $\mathcal{D}^n_{ss}$ for $j \in J$, then U is of type $\mathcal{D}^n_{ss}$ by Corollary 5.3. By Proposition 5.13 (and Lemma 2.5) U is also concave; applying Proposition 4.5.2 completes the proof. □

Because equilibrium existence is preserved under unimodular basis changes (the clarity of this is one of the benefits of our representation of demand), an elementary application of Proposition 5.13 is:

Corollary 5.15 (cf. Sun and Yang, 2006, Theorem 3.1)**.** *Equilibrium always exists for the 'generalised gross substitutes and complements' type of demand.*

Proof: Immediate from Propositions 5.13, 4.14 and 5.5. □

Note this corollary also provides another proof of Example 5.12's "even cycle of complements" result. If we separate the goods into two classes corresponding to the odd- and even-numbered goods, and re-order so that all the odd ones come first, demand is then of type $\mathcal{D}_{GGSC}^{n/2,n/2}$, so Corollary 5.15 applies.

Moreover, we can now generalise further to an even *more* general style of GGSC-like demand, in which goods are separated into an arbitrary number of groups, with goods within the same group being strong substitutes, but with 1-1 complementarities between some pairs of groups (that is, for those pairs of groups, each good in one of the groups may exhibit 1-1 complementarities with any good in the other group). If all the "cycles" formed by the sequences of "paired" groups are of even length, then we can again separate the groups of goods into two classes, so that the demand is again GGSC demand, and so always has a competitive equilibrium. But if any odd cycle exists then, just as in Example 5.12, competitive equilibrium may fail.[42]

5.3.3 When is Strong Substitutes a necessary condition for equilibrium?

By enumerating possible sub-cases it is not too hard to show (see Appendix B.4) that if there are at most three goods, equilibrium can always exist *only* if demand is a (unimodular) basis change from strong substitutes. So:

Theorem 5.16. *In $\mathbb{R}^3$, equilibrium always exists for a concave demand type $\mathcal{D}$ if and only if it is a unimodular basis change from strong substitutes, or a subset thereof.*

One might therefore wonder whether equilibrium always exists only when demand is a basis change from strong substitutes. However, we now show that this is *not* the case, by exhibiting a 4-D demand type which is genuinely different from strong substitutes, and always has a competitive equilibrium. Consider the demand type defined by the matrix

$$D := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We show this always has a competitive equilibrium in Appendix B.4.

To show it is not just a basis change from 4-D strong substitutes, $\mathcal{D}_{ss}^4$, assume (for contradiction) there exists a unimodular matrix G such that $G^T D$ consists entirely of distinct column vectors from $\mathcal{D}_{ss}^4$.[43] Since D has 9 columns, $G^T D$ must include all but one of the 10 distinct vectors in $\mathcal{D}_{ss}^4$. Note that every row $\mathbf{r}$ of D satisfies $\mathbf{r}.\mathbf{w} = 0$, where $\mathbf{w} = (1, 1, 1, 1, 1, 1, -1, -1, -1)$, so the rows $\mathbf{r}'$ of $G^T D$ must also satisfy $\mathbf{r}'.\mathbf{w} = 0$ (since pre-multiplying D by any matrix generates a new matrix whose rows are linear

[42]This result has independently been established by Sun and Yang (private communication), and also Teytelboym (2012); the latter paper gives fuller details.

[43]Recall that vectors which are the negation of one another are *not* considered "distinct" in our framework.

combinations of D's rows). But there are precisely four vectors in $\mathcal{D}^4_{ss}$ with non-zero entry in any coordinate i ($\mathbf{e}^i$, and $\mathbf{e}^i - \mathbf{e}^j$ for the three values of $j \neq i$), so there are four non-zero entries in every row of the matrix whose columns are the 10 distinct vectors of $\mathcal{D}^4_{ss}$, and if we delete any one column, then at least one row must have exactly three non-zero entries. Since these three entries are ± 1, there is no way to add or subtract the three together to obtain zero; it is impossible that this row has zero dot product with $\mathbf{w}$. Thus no 9 vectors of $\mathcal{D}^4_{ss}$ can form the columns of $G^T D$, for any unimodular matrix G.[44]

5.4 Existence of equilibrium for specific demands

Our Theorem 5.6 tells us which demand types *always* have a competitive equilibrium. When the answer is negative, it does not tell us whether competitive equilibrium exists for every supply bundle, for a specific set of demands. But if all intersections are "nice" (in the sense of Section 5.2) then we can apply Proposition 5.10 to each intersection point to check for such a failure.

Take, for example, Agents 1 and 2 who have THs of the combinatorial types of Figures 1 and 11, respectively, and concave valuations. (A valuation function of the combinatorial type of Figure 1 *must* be concave. A valuation function of the type of Figure 11 need not be concave, though the specific valuation function of this type that is given in Example 2.9 is concave.)

The combinatorial type of aggregate demand will depend on how the agents' THs meet in price space; assume they only intersect "nicely". Applying Propositions 5.9 and 5.10, we see that there exists a supply bundle such that competitive equilibrium does not exist iff the facets with normals $(1, 0)$ and $(-1, 2)$ intersect (since $\det \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} = 2 > 1$). An example of aggregate demand of this combinatorial type is illustrated in Figure 18a; the bundle $(1, 1)$ is in the interior of the parallelogram in the SNP of Figure 18a, and is never demanded on aggregate (see Proposition 5.9).

Combinatorial types of aggregate demand in which competitive equilibrium *does* exist for any supply bundle are illustrated in Figures 18b, 18c and 18d (there are others).

In Figures 18b and 18c, there are *two* intersections between the THs. In each case, the areas of the SNP faces corresponding to the intersections are 1. We call this area the 'multiplicity' of the intersection; note that it is, of course, the determinant of the (primitive integer) edges of the SNP face (and so, as we have seen, intimately connected with the existence of competitive equilibrium).

Conversely, in Figures 18a and 18d there is only *one* intersection. Now, however, the corresponding SNP face has area 2; we say the 'multiplicity' of the intersection is 2.

Observe that in each case, the number of intersections, weighted by multiplicity, is 2. It can be checked that this holds for every other aggregate of the demands of two

[44] What might this demand represent? The final good is worth nothing on its own (the vector $\mathbf{e}^4$ is *not* in $\mathcal{D}$) but increases the value of any of the first three goods, as shown by the pairwise complementarities that form the 4th to 6th columns of D. Furthermore, there are pairwise complementarities between the first three goods *only* in the additional presence of the fourth good: this is the meaning of the final three columns. So the first three goods might be front-line workers, and the fourth a facilitator or manager.

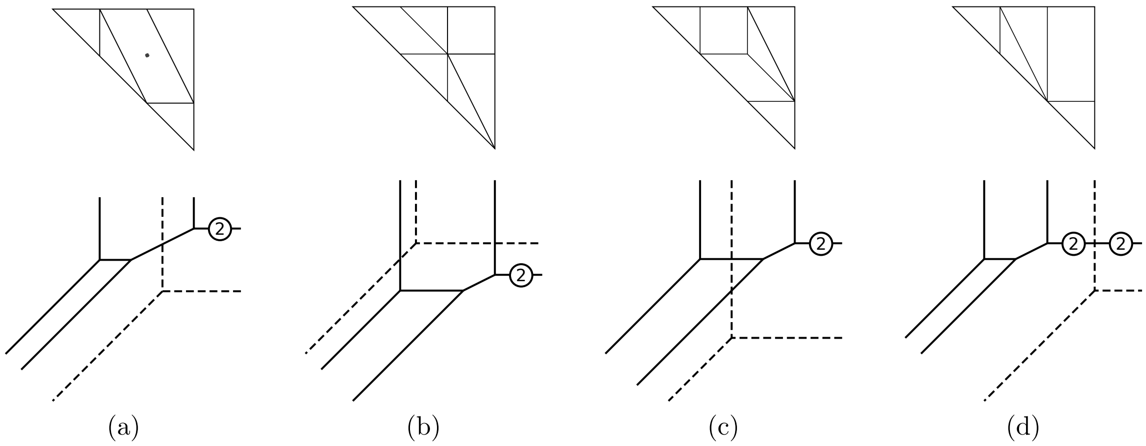


Figure 18: Examples of aggregate THs and SNPs of agents with THs of the combinatorial types of Fig. 1 (dashed line) and Fig. 11 (solid line). The number of intersections of the THs, weighted by facet weights, reveals the existence or failure of equilibrium.

agents whose individual THs are of the same combinatorial types as Figures 1 and 11, respectively. This is a special case of the *Tropical Bézout Theorem*.[45]

However, the natures of the multiplicity 2 intersections in Figures 18a and 18d are different. In Figure 18d, one of the corresponding facets is of weight 2; Agent 2 has a concave valuation and so has *three* bundles in its demand set, so Proposition 5.9 does not apply–the bundles 'inside' the weight-2 facet (in the centres of the long edges of the rectangle in the SNP) are both demanded at this price. The best way to understand this situation is that 'two intersections have become arbitrarily close'. By contrast, in Figure 18a, neither of the corresponding facets has weight 2, Proposition 5.9 *does* apply, and the bundle in the centre of the parallelogram is *not* demanded at any price.

Recall that the multiplicity of the intersection is the area of the SNP face, which equals the (absolute value of the) determinant of its edges. The key point is that this can be factorised into the product of the facet weights times the (absolute value of the) determinant of the *primitive integer* edge directions (that is, the primitive integer facet normals). And equilibrium fails iff the (absolute value of the) *latter* determinant exceeds 1. So the *existence* of a supply bundle for which competitive equilibrium fails is signalled by a case in which the sum of intersections, weighted *only* by facet weights, is too small.

These ideas can be applied more generally, as will be developed in future work.

6 Conclusion

Studying the tropical geometry of demand yields a range of insights. The structure of an agent's preferences can be efficiently summarised by a set of vectors that is orthogonal to the divisions between the regions of price space in which the agent demands different bundles. The same set of vectors also generates the surface of the convex hull of the agent's valuation function in quantity space. The duality between these representations

[45]See Richter-Gebert et al (2005).

has powerful implications, and the pictorial representations that tropical geometry gives us generate new intuitions.[46]

We began this work while studying the properties of many-dimensional Product-Mix Auctions. Convex and tropical geometry is the key to much of our analysis in Baldwin and Klemperer (in preparation) in which we describe ways in which different preferences can be represented in these auctions, and the implications of different restrictions on bids.[47] Geometric reasoning has also helped us develop extensions to the Bank of England's original implementation of the auction,[48] and understand the connections to related auction designs.[49]

Our current paper has shown that convex- and tropical-geometric analysis provides an efficient way of determining the "type" of demand, and it has also proven a new theorem about the existence of competitive equilibrium.

In other work, we have found that similar geometric analysis is useful in understanding results obtained by others, and that it can prove these results more quickly than currently-used techniques. So we are optimistic that tropical-geometric analysis will yield more economic insights in the future; we hope others will take up these methods.

A Standard concepts of convex geometry

[This section to be completed later]

B Proofs of Results in the text

B.1 Proofs of Results in Section 4

Proof of Proposition 4.2 1. By definition, $\mathbf{x} \in D_u(\mathbf{p})$ if $\mathbf{p}^T(\mathbf{x} - \mathbf{x}') \leq u(\mathbf{x}) - u(\mathbf{x}')$ for all $\mathbf{x}' \in A$, with equality iff $\mathbf{x}' \in D_u(\mathbf{p})$ also. For any invertible matrix G, we may re-write

$$\mathbf{p}^T(\mathbf{x} - \mathbf{x}') = \mathbf{p}^T G G^{-1}(\mathbf{x} - \mathbf{x}') = (G^T\mathbf{p})^T(G^{-1}\mathbf{x} - G^{-1}\mathbf{x}').$$

[46]These intuitions are obscured by existing pictorial representations which shoehorn indivisible demand into the standard divisible-demand framework.

[47]In the Bank of England's implementation, the bid-taker expresses preferences through a "supply function" while bidders can make sets of "or" bids that can, if desired, be represented as sets of points on a graph. Permitting negative as well as positive bids broadens the set of preferences that can be expressed, as does permitting bidders to specify additional constraints (Klemperer, 2008, 2010). The issue is: what kinds of bids should we permit to achieve a sufficiently rich representation of preferences, while retaining a unique solution (the extent to which we can permit some degree of complements is a particular challenge), achieving an efficient outcome (in particular, not incentivising strategic behaviour), and retaining simplicity and transparency?

[48]Extensions include broadening the range of contexts to which these (or related) auctions can be applied, through a better understanding of when equilibrium is guaranteed to exist, as well as better ways of representing bidders' and bid-takers' multi-dimensional preferences.

[49]Related designs include, in particular, the Assignment Auction suggested independently by Milgrom (2009), and versions of Simultaneous Multiple Round Auction (see, e.g., Milgrom, 2000) and "Clock Auctions" (see, e.g., Ausubel and Milgrom, 2002, Gul and Stacchetti, 2000, and Milgrom and Strulovici, 2009); see also the papers in Cramton, Shoham, and Steinberg (2006). As noted in the Introduction, we are also concerned with efficient solution techniques for Product-Mix Auctions, both when we need integer solutions, and when rationing is permitted, etc.

If G is additionally unimodular, then $G^{-1}\mathbf{x}$ and $G^{-1}\mathbf{x}' \in \mathbb{Z}^n$. We define a new valuation G^*u on the finite set $G^{-1}A \subsetneq \mathbb{Z}^n$ via $G^*u(\mathbf{y}) := u(G\mathbf{y})$. If we write $\mathbf{y} = G^{-1}\mathbf{x}$ and $\mathbf{y}' = G^{-1}\mathbf{x}'$ then $(G^T\mathbf{p})^T(\mathbf{y}-\mathbf{y}') \leq G^*u(\mathbf{y}) - G^*u(\mathbf{y}')$ holds iff $\mathbf{p}^T(\mathbf{x}-\mathbf{x}') \leq u(\mathbf{x}) - u(\mathbf{x}')$. So we have

$$\mathbf{x} \in D_u(\mathbf{p}) \;\Leftrightarrow\; \mathbf{y} = G^{-1}\mathbf{x} \in D_{G^*u}(G^T\mathbf{p}),$$

as required.

2. Since the underlying set of $\mathcal{T}_u$ is those $\mathbf{p}$ for which $\#D_u(\mathbf{p}) > 1$ it follows immediately from 1. that $\mathcal{T}_{G^*u} = \{G^T\mathbf{p} \mid \mathbf{p} \in \mathcal{T}_u\}$, as required.

3. It follows from 2. that if $\mathbf{v}$ is normal to a facet of $\mathcal{T}_u$ then $G^T\mathbf{v}$ is normal to a facet of $\mathcal{T}_{G*u}$. As G has an integer inverse, the converse is also true. Trivially, for any unimodular matrix G, the valuation G^*u is concave iff the valuation u is. $\square$

Proof of Proposition 4.5: 1. Suppose an agent's valuation is not of type $\mathcal{D}^n_{os}$. Then its TH has a facet, F, with a primitive integer normal $\mathbf{v}$ which has entries v_i and v_j which are both positive.[50] So some integer bundle $\mathbf{x}$ is demanded on one side, and on the other, the bundle $\mathbf{x} + m\mathbf{v}$ is demanded, for some $m \in \mathbb{Z}_{>0}$. Pick a price $\mathbf{p}$ in the interior of F. Note that the vector $\mathbf{e}^i$ does not lie within the facet, as $\mathbf{e}^i.\mathbf{v} = v_i > 0$. So there exists $\epsilon > 0$ such that $\mathbf{p} - \epsilon\mathbf{e}^i$ and $\mathbf{p} + \epsilon\mathbf{e}^i$ are in the UDRs on either side of F. So increasing the price of good i from $\mathbf{p} - \epsilon\mathbf{e}^i$ to $\mathbf{p} + \epsilon\mathbf{e}^i$ changes the set of demands from $\{\mathbf{x}+m\mathbf{v}\}$ to $\{\mathbf{x}\}$ (since buying additional units of good i is now slightly less attractive). But $mv_j > 0$, so an increase in the price for good i has reduced demand for good j. Thus demand is not ordinary substitutes.

Now suppose that u is of type $\mathcal{D}^n_{os}$. Consider $\mathbf{p}' \geq \mathbf{p}$ such that $D_u(\mathbf{p})$ and $D_u(\mathbf{p}')$ contain only one bundle. Generically the line $[\mathbf{p}, \mathbf{p}']$ crosses only facets, not any lower dimensional cells in $\mathcal{T}_u$. Furthermore, because the UDRs are open sets and because there are only finitely many cells of lower dimension than $n-1$, we can chose always prices $\mathbf{q}$ and $\mathbf{q}'$ such that the only bundle demanded at $\mathbf{q}$ is the bundle demanded at $\mathbf{p}$, the only bundle demanded at $\mathbf{q}'$ is the bundle demanded at $\mathbf{p}'$, and the line $[\mathbf{q}, \mathbf{q}']$ is in the same direction as $[\mathbf{p}, \mathbf{p}']$ and does cross only facets in $\mathcal{T}_u$. All the facets have normal vector in $\mathcal{D}^n_{os}$ and so crossing one corresponds to demanding less of at most one good, and more of at most one good. By the strict law of demand (which applies since utility is quasilinear), the good of which less is demanded must be a good whose price is changing, and so the quantity of those whose price does not change is weakly increased. As this applies to the pair $[\mathbf{q}, \mathbf{q}']$, it also applies to $[\mathbf{p}, \mathbf{p}']$.

2.[51] This proof relies on the following result (Baldwin and Klemperer 2012, Proposition B.1, which is a slight extension of Milgrom and Strulovici 2009, Theorem 12): a valuation $u : A \to \mathbb{R}$ is a strong substitute valuation iff it is an ordinary substitute valuation and satisfies the consecutive integer property, that is, for any $\mathbf{p} \in \mathbb{R}^n$ and for $i = 1, \ldots, n$, the set $\{x_i \mid \mathbf{x} \in D_u(\mathbf{p})\}$ consists of consecutive integers.

First assume that u is a strong substitute valuation; as above, it follows that it is an ordinary substitute valuation and satisfies the consecutive integer property. By 1. above, it has demand type $\mathcal{D}^n_{os}$. But if $\alpha\mathbf{e}^i - \beta\mathbf{e}^j$ is a facet normal in $\mathcal{D}^n_{os}$, but not in $\mathcal{D}^n_{ss}$, then $\alpha, \beta > 0$ are coprime and not both equal to 1, and so, at any price on the interior of this facet, the consecutive integer property is violated. So u is of demand type $\mathcal{D}^n_{ss}$.

[50] If $\mathbf{v}$ is a facet normal then so is $-\mathbf{v}$ so this assumption is without loss of generality.

[51] In the case that $n = 1$, Proposition 4.5.2 follows from Kelso and Crawford Theorem 6.

Finally, u is concave (Milgrom and Strulovici 2009, Theorem 9).

On the other hand, suppose u is of concave type $\mathcal{D}^n_{ss}$. By Part 1. we know it is an ordinary substitute valuation, so it is sufficient to show that it satisfies the consecutive integer property. For any price $\mathbf{p}$, consider the SNP face $D_u(\mathbf{p})$. For any good i, the sets of bundles in $D_u(\mathbf{p})$ minimal and maximal for i must contain vertices of $D_u(\mathbf{p})$. The set of vertices of $D_u(\mathbf{p})$ is connected by the set of edges of $D_u(\mathbf{p})$, and so there exists a path along edges from a minimal vertex for i to a maximal vertex for i; all lattice points along this path are in $D_u(\mathbf{p})$. But every edge of $D_u(\mathbf{p})$ is some integer multiple of a vector $\mathcal{D}^n_{ss}$; since u is concave, every intermediate integer multiple of this vector is also in $D_u(\mathbf{p})$. So we may take a path from a vertex of $D_u(\mathbf{p})$ minimal with respect to i to one maximal with respect to i, along only vectors in $\mathcal{D}^n_{ss}$. But the non-zero coordinate entries of these vectors are ± 1. Thus the quantity of good i demanded changes by at most 1 at each step; this demonstrates the consecutive integer property. Thus u is a strong substitute valuation, as required. $\square$

Proof of Proposition 4.8: (This proof is similar to that of Proposition 4.5.1.) Suppose an agent's valuation is not of type $\mathcal{D}^n_c$. Then its TH has a facet, F, with a primitive integer normal $\mathbf{v}$ which has entries v_i and v_j where $v_i > 0$ and $v_j < 0$. So some integer bundle $\mathbf{x}$ is demanded on one side, and on the other, the bundle $\mathbf{x} + m\mathbf{v}$ is demanded, for some $m \in \mathbb{Z}_{>0}$. Pick a price $\mathbf{p}$ in the interior of F. Note that the vector $\mathbf{e}^i$ does not lie within the facet, as $\mathbf{e}^i.\mathbf{v} = v_i > 0$. So there exists $\epsilon > 0$ such that $\mathbf{p} - \epsilon\mathbf{e}^i$ and $\mathbf{p} + \epsilon\mathbf{e}^i$ are in the UDRs on either side of F. So increasing the price of good i from $\mathbf{p} - \epsilon\mathbf{e}^i$ to $\mathbf{p} + \epsilon\mathbf{e}^i$ changes the set of demands from $\{\mathbf{x} + m\mathbf{v}\}$ to $\{\mathbf{x}\}$ (since buying additional units of good i is now slightly less attractive). But $mv_j < 0$, so an increase in the price for good i has increased demand for good j. Thus demand is not complements.

Now suppose that u is of type $\mathcal{D}^n_c$. For some $i > 0$, consider $\mathbf{p}' = \mathbf{p} + \delta\mathbf{e}^i$ for some $\delta > 0$, such that $D_u(\mathbf{p})$ and $D_u(\mathbf{p}')$ contain only one bundle. Following exactly the argument given in the proof of Proposition 4.5.1, we can assume that the line $[\mathbf{p}, \mathbf{p}']$ crosses only facets. As the price increases from $\mathbf{p}$ to $\mathbf{p}'$, demand for good i weakly decreases by revealed preference. But since every facet normal is in $\mathcal{D}^n_c$ it follows that demand for every other good must also weakly decrease. Since we may break down any price increase from $\mathbf{p}$ to $\mathbf{p}' \geq \mathbf{p}$ into a series of increases in a single price, this completes the proof. $\square$

Proof of Proposition 4.10. Recall that if demand is additively separable then a change in the price p_i for any one good has no effect on the demand for other goods.

Consider $\mathcal{T}_u$ not of demand type $\mathcal{D}^n_a$. Then $\mathcal{T}_u$ has a facet F whose normal $\mathbf{v}$ has two non-zero entries, v_i and v_j (where $i \neq j$). We may cross this facet by changing only price p_i; this has a non-zero effect on the demand for good j. So u is not additively separable in this case.

Now suppose u is of concave demand type $\mathcal{D}^n_a$. It follows that the only SNP edges are the coordinate vectors, and that every integer bundle within an SNP face is valued at the appropriate convex combination of the values of the vertices of the face. But this implies that the value of an additional unit of a good is independent of the number of units of other goods one possesses, which is additive separability. $\square$

B.2 Proofs of Results in Section 5.1

Proof of Propositions 5.1 and 5.2. Proposition 5.1 is straightforward. Note that

$$\sum_{j\in J}\max_{\mathbf{x}^j\in A}\{u^j(\mathbf{x}^j)-\mathbf{p}.\mathbf{x}^j\}=\max\left\{\sum_{j\in J}u^j(\mathbf{x}^j)-\mathbf{p}.\left(\sum_{j\in J}\mathbf{x}^j\right)\ \mid \mathbf{x}^j\in A^j, j\in J\right\},$$

and on the other hand (since $\mathbf{y}\in A$ iff $\mathbf{y}=\sum_{j\in J}\mathbf{x}^j, \mathbf{x}^j\in A_J$) that

$$\begin{aligned}
&\max_{\mathbf{y}\in A}\{U(\mathbf{y})-\mathbf{p}.\mathbf{y}\}\\
&= \max\left\{\max\left\{\sum_{j\in J}u^j(\mathbf{x}^j)\mid \mathbf{x}^j\in A_j, \sum_{j\in J}\mathbf{x}^j=\mathbf{y}\right\}-\mathbf{p}.\mathbf{y}\mid \mathbf{y}=\sum_{j\in J}\mathbf{x}^j, \mathbf{x}^j\in A_j j\in J\right\}\\
&= \max\left\{\sum_{j\in J}u^j(\mathbf{x}^j)-\mathbf{p}.\left(\sum_{j\in J}\mathbf{x}^j\right)\ \mid \mathbf{x}^j\in A, j\in J\right\},
\end{aligned}$$

and that the same arguments $\mathbf{x}^j\in A$, with $\mathbf{y}=\sum_{j\in J}\mathbf{x}^j$, are maximising in either case.

The text showed the underlying sets of $\mathcal{T}_U$ and $\mathcal{T}_{\{u^j\}}$ are the same, so completing the proof of Proposition 5.2 only requires checking the weightings are the same. So suppose F is a facet of $\mathcal{T}_U$ with adjacent UDRs U and U'; let $\mathbf{v}_F$ be a primitive integer vector pointing from U to U'. Suppose agent j demands $\mathbf{x}^j$ in U and $\mathbf{x}^{j\prime}$ in U' (for some agent these will be distinct, but not necessarily for all). Then $w_j(F)\mathbf{v}_F=\mathbf{x}^{j\prime}-\mathbf{x}^j$ for all j, and so

$$\sum_j w_j(F)\mathbf{v}_F=\sum_j\mathbf{x}^{j\prime}-\sum_j\mathbf{x}^j.$$

So $w_U(F)=\sum_j w_j(F)=w_{\{u^j\}}(F)$, as required. □

Proof of Proposition 5.5. Suppose $G^T\mathcal{D}$ always has a competitive equilibrium. Consider any agent valuations $u^1,\ldots,u^k$ of type $\mathcal{D}$ and let $\mathbf{x}$ be in the support of their aggregate valuation. Then demands $G^*u^1,\ldots,G^*u^k$ have type $G^T\mathcal{D}$ and $\mathbf{y}:=G^{-1}\mathbf{x}$ is in their aggregate valuation set. By assumption competitive equilibrium exists in the latter case: there exists a price $\mathbf{p}$ at which the agent with valuation G^*u^i demands $\mathbf{y}^i$ and $\sum_i\mathbf{y}^i=\mathbf{y}$. But then in each case we may define $\mathbf{x}^i:=G\mathbf{y}^i\in D_{u^i}(G^{-T}\mathbf{p})$ (see Proposition 4.2.1). At price $G^{-T}\mathbf{p}$ the market clears for $\mathbf{x}:=\sum_i\mathbf{x}^i$. So $\mathcal{D}$ has a competitive equilibrium. The converse is shown by repeating the argument, using the unimodular matrix G^{-T}. □

B.3 Proof of results in Section 5.2

This Appendix gives the additional details needed to complete the proof of Theorem 5.6. Lemmas B.1, B.4 and B.5 demonstrate that generically all single-point intersections of the TH are "nice". The logic is as follows: first (Lemma B.1), we show how to perform affine translations of agents' THs, and bound the associated change in valuation. Now consider an intersection of two cells from distinct agents' THs. Generically (in the space of affine translations) there can be no vector normal to both; if there were, a small shift the agents' demands in the direction of this vector would mean the cells no longer intersected at all. We argue thus in Lemma B.4.

In Lemma B.5 we show how to make all intersections 'nice', while bounding the change in any agent's valuation. Begin by considering an intersection of two cells from distinct agents' THs. Generically, there can be no vector normal to both, since if there were, a small shift of one of the agents' demands in the direction of this vector would mean the cells no longer intersected at all. Make such a shift, if necessary. Now, for each of the two cells that intersect, we nominate a linearly independent set of vectors normal to adjacent facets. The fact that there is no vector normal to both the cells means that the union of these sets remains linearly independent. But the intersection of the two cells is now a cell of the TH of the aggregate demand of the two agents, and the collection of vectors we have defined so far are normal to facets in this TH whose intersection is this new cell. Continuing to add any additional agents' demands that intersect the cell generically, we can construct a set of linearly independent vectors, each normal to a facet of the TH of aggregate demand, such that the intersection of these facets locally defines the intersection of the cells in question.

It follows that, if the equivalent conditions of Remark 5.8 are satisfied, we may apply Proposition 5.10 at any intersection of agents' THs. So, after these small perturbations, any bundle is demanded at some price. We complete the proof of Theorem 5.6 by showing that, if a bundle is demanded following an extremely small perturbation in agents' valuations, it must have also been demanded before this perturbation. This proves the sufficiency of the condition given in Theorem 5.6: that any linearly independent subset of vectors in the demand type are an integer basis for their span.

Necessity of this has already been provided by Proposition 5.9.

First, then, we introduce the affine perturbations discussed above.

Lemma B.1. *Suppose an agent has valuation function $u : A \to \mathbb{R}$. For any $\mathbf{w} \in \mathbb{R}^n$, we may define a valuation function $u_{\mathbf{w}} : A \to \mathbb{R}$ such that, for all $\mathbf{p} \in \mathbb{R}^n$, we have*

1. $D_{u_{\mathbf{w}}}(\mathbf{p}) = D_u(\mathbf{p} + \mathbf{w})$;
2. $\mathcal{T}_{u_{\mathbf{w}}} = \{\mathbf{p} - \mathbf{w} \mid \mathbf{p} \in \mathcal{T}_u\}$;
3. $\|u_{\mathbf{w}}(\mathbf{x}) - u(\mathbf{x})\| \leq R\|\mathbf{w}\|$, *where R satisfies $\|\mathbf{x}\| < R$ for all $\mathbf{x} \in A$.*

Proof. Let $u_{\mathbf{w}}(\mathbf{x}) = u(\mathbf{x}) - \mathbf{x}.\mathbf{w}$. Then

$$D_{u_{\mathbf{w}}}(\mathbf{p}) = \arg\max_{\mathbf{x}\in A}\{u(\mathbf{x}) - \mathbf{x}.\mathbf{w} - \mathbf{x}.\mathbf{p}\} = \arg\max_{\mathbf{x}\in A}\{u(\mathbf{x}) - \mathbf{x}.(\mathbf{p} + \mathbf{w})\} = D_u(\mathbf{p} + \mathbf{w}).$$

The remainder of the lemma follows by definition of $\mathcal{T}_u$, and the CauchySchwarz inequality. $\square$

To prove that the hypotheses of Proposition 5.10 are satisfied after such perturbations, is convenient to use "annihilator spaces". For a linear or affine subspace of $\mathbb{R}^n$, these give the linear subspace of all orthogonal vectors. We recall their definition and basic properties.

Definition B.2 (See e.g. Spence et al., 2000)**.** If $C \subseteq \mathbb{R}^n$ is an affine subspace, define

$$C^\circ := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}.(\mathbf{c} - \mathbf{c}') = 0,\ \forall \mathbf{c}, \mathbf{c}' \in C\}.^{52}$$

Note that if $D = C + \mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^n$ then $D^\circ = C^\circ$.

We use annihilator spaces for the following results.

Lemma B.3 (See e.g. Spence et al., 2000)**.** *Suppose that $C_1, C_2 \subseteq \mathbb{R}^n$ are affine subspaces.*

1. *If $C_1 \subseteq C_2$ then $C_2^\circ \subseteq C_1^\circ$*
2. *If $C_1 \cap C_2 \neq \emptyset$ then additionally $(C_1 \cap C_2)^\circ = C_1^\circ + C_2^\circ$.*
3. $\dim C_1 + \dim(C_1)^\circ = n$

Proof. Part 1 is clear. Part 2 follows from the standard result when C_1 and C_2 are linear subspaces (see, e.g. Spence et al. 2000): if $-\mathbf{w} \in C_1 \cap C_2$ then $C_1 + \mathbf{w}$ and $C_2 + \mathbf{w}$ are linear subspaces, and so $((C_1 + \mathbf{w}) \cap (C_2 + \mathbf{w}))^\circ = (C_1 + \mathbf{w})^\circ + (C_2 + \mathbf{w})^\circ$. But $(C_1 + \mathbf{w}) \cap (C_2 + \mathbf{w}) = (C_1 \cap C_2) + \mathbf{w}$ so the result follows from the note above. Part 3 similarly follows immediately from the linear case. □

Now we show that any two THs may be perturbed so that the intersection of their cells is 'generic' (as given in the statement of the following lemma):

Lemma B.4. *Suppose we have agents* 1 *and* 2 *with valuation functions u^1 and u^2 (not necessarily concave). For any $\epsilon > 0$ we may find a vector $\mathbf{w}$ such that, if we perturb agent 2's demand by $\mathbf{w}$ to obtain $u^2_{\mathbf{w}}$, then $\|u^2_{\mathbf{w}}(\mathbf{x}) - u^2(\mathbf{x})\| < \epsilon$ for all $\mathbf{x} \in A$, and any cells C_1 of $\mathcal{T}_{u^1}$ and $C_2^{\mathbf{w}}$ of $\mathcal{T}_{u^2_{\mathbf{w}}}$ satisfy $C_1 \cap C_2^{\mathbf{w}} \neq \emptyset \Rightarrow C_1^\circ \cap (C_2^{\mathbf{w}})^\circ = \{0\}$.*

Proof. Suppose that C_1 in $\mathcal{T}_{u^1}$ and C_2 in $\mathcal{T}_{u^2}$ satisfy $C_1 \cap C_2 \neq \emptyset$ and $C_1^\circ \cap C_2^\circ \neq \{0\}$. Choose $\mathbf{w}_1 \in C_1^\circ \cap C_2^\circ$ with $\mathbf{w}_1 \neq 0$. Then, for all $\eta > 0$, we show that $(C_2 + \eta\mathbf{w}_1) \cap C_1 = \emptyset$. For, given any $c_2 \in C_2$, if $c_1 \in C_1 \cap C_2$ then $\mathbf{w}_1.(c_1 - (c_2 + \eta\mathbf{w}_1)) = \eta\|\mathbf{w}_1\|^2 \neq 0$ (since $c_1, c_2 \in C_2$) and so, since $\mathbf{w}_1 \in C_1^\circ$, it follows that $c_2 + \eta\mathbf{w}_1 \notin C_1$.

On the other hand, recall that the cells of THs are closed objects. It follows that a sufficiently small perturbation of one of the THs will not introduce any new intersections between cells. So there exists $\eta_1 > 0$ such that if $\eta < \eta_1$ then no new intersections arise.[53]

Since THs consist of a finite number of affine cells, we may suppose that there are in total d intersections of cells in $\mathcal{T}_{u^1}$ and $\mathcal{T}_{u^2}$ whose annihilator spaces have non-zero intersection. We find $\mathbf{w}_j$ and η_j as above for each in turn, and apply them all.[54] Thus, perturbing Agent 2 by $\mathbf{w} = \eta\mathbf{v}$, where $\mathbf{v} = \sum_{j=1}^{d} \eta_j \|\mathbf{w}_j\|$ and $\eta \in (0, 1]$, gives us the intersection properties required. To ensure that the perturbation to the agent's valuation is sufficiently small, we choose $\eta < \frac{\epsilon}{R\|\mathbf{v}\|}$ where R satisfies $\|\mathbf{x}\| < R$ for all $\mathbf{x} \in A$. By Lemma B.1.3, this implies that $\|u^2_{\mathbf{w}}(\mathbf{x}) - u^2(\mathbf{x})\| < \epsilon$ for all $\mathbf{x} \in A$, as required. □

[52] If $D \leq \mathbb{R}^n$ is a linear subspace then this definition clearly coincides with the usual $D^\circ := \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{v}.\mathbf{d} = 0\ \forall \mathbf{d} \in D\}$.

[53] To be precise: if $C_{2a}^{\mathbf{w}}$ in $\mathcal{T}_{u^2_{\mathbf{w}}}$ satisfies $C_{2a}^{\mathbf{w}} \cap C_{1a} \neq \emptyset$ for any cell C_{1a} in $\mathcal{T}_{u^1}$ then the corresponding C_{2a} in $\mathcal{T}_{u^2}$ satisfies $C_{2a} \cap C_{1a} \neq \emptyset$.

[54] Strictly speaking, each η_j should be found when we compare the cells after $\mathcal{T}_{u^2}$ has undergone the translations corresponding to intersections $1, \ldots, j-1$.

We may now take a set of m agents, and shift each so that their valuation for any bundle is changed by at most ϵ, and nearly all the conditions of Proposition 5.10 are met at every intersection of the THs. The only condition we do not insist on is that the set of primitive integer facet normals are an integer basis for their span; whether or not this could possibly hold will depend on the demand types of the agents in question. What we prove is that these vectors are linearly independent.

Lemma B.5. *Suppose we have m agents, with valuations u^i for $i = 1, \ldots, m$. For every $\epsilon > 0$ we may perturb each agent's valuation by a vector $\mathbf{w}^i$ such that $\|u^i(\mathbf{x}) - u^i_{\mathbf{w}^i}(\mathbf{x})\| < \epsilon$ for all $\mathbf{x}$ in $\mathbb{R}$, and such that, whenever a price point $\mathbf{p}$ is in the interior of $(n - k_{i_j})$-cell C_{i_j} of the TH $\mathcal{T}_{u^{i_j}}$ for agents $i_1, \ldots, i_s$, then each C_{i_j} is locally to $\mathbf{p}$, given by the intersection of a set of facets $F_1^{i_j}, \ldots, F_{k_{i_j}}^{i_j}$ of $\mathcal{T}_{u^{i_j}}$ (not necessarily comprising all facets of $\mathcal{T}_{u^{i_j}}$ that pass through C_{i_j}) with primitive integer normal vectors $\mathbf{v}_1^{i_j}, \ldots \mathbf{v}_{k_{i_j}}^{i_j}$, such that the full set $\{\mathbf{v}_l^{i_j} \mid j = 1, \ldots, s; l = 1, \ldots, k_{i_j}\}$ is linearly independent.*

Proof. We make a series of perturbations of agents' individual demands, as in Lemmas B.1 and B.4. First, we allow Agent 1 to remain unperturbed. For $i = 2, \ldots, m$ we compare:

1. the TH of aggregate demand of agents $1, \ldots, i - 1$;

2. the TH of agent i.

In each case, we apply Lemma B.4 to find $\mathbf{w}^i$ with $\|u^i(\mathbf{x}) - u^i_{\mathbf{w}^i}(\mathbf{x})\| < \epsilon$, and such that, after the perturbation, $C_i^\circ \cap C^\circ = \{0\}$ whenever $C_i \cap C \neq \emptyset$, where C_i is any cell in $\mathcal{T}_{u^i}$ and C is any cell in the TH of aggregate demand of agents $1, \ldots, i - 1$.

Write U' for the new aggregate demand, after all agents have been perturbed. Now we need to see that the hypotheses of Proposition 5.10 are satisfied at every intersection of individual perturbed THs that make up $\mathcal{T}_{U'}$. Consider a price point $\mathbf{p}$, which lies in the interior of $(n - k_{i_j})$-cells $C_{i_1}, \ldots, C_{i_s}$ of the THs of individual demand from distinct agents $i_1, \ldots, i_s$ respectively, where we index so that $i_1 < \cdots < i_s$. From Lemma B.3.3 we know that $\dim C_{i_j}^\circ$ is k_{i_j}.

Let $C := \bigcap_{j=1}^s C_{i_j}$. By Lemma B.3.2, we know that $C^\circ = \left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^\circ + C_{i_s}^\circ$. On the other hand, $\mathbf{p} \in \bigcap_{j=1}^{s-1} C_{i_j}$, and so there is a cell C' of the tropical variety of aggregate demand of agents $1, \ldots, i_s - 1$, with $\mathbf{p} \in C'$. Since demand is constant in the interior of a cell, it follows that $C' \subseteq C_{i_j}$ for $j = 1, \ldots, i_{s-1}$ and so $C' \subseteq \bigcap_{j=1}^{s-1} C_{i_j}$. We know that $\mathbf{p} \in C_{i_s} \cap C'$ and so, by the construction of the perturbations, we know $C'^\circ \cap C_{i_s}^\circ = \{0\}$. As $C' \subseteq \bigcap_{j=1}^{s-1} C_{i_j}$, it follows by Lemma B.3.1 that $\left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^\circ \subseteq C'^\circ$, so we may conclude that $\left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^\circ \cap C_{i_s}^\circ = \{0\}$. Thus

$$C^\circ = \left(\bigcap_{j=1}^{s-1} C_{i_j}\right)^\circ \oplus C_{i_s}^\circ.$$

Proceeding inductively

$$C^\circ = C_{i_1}^\circ \oplus \cdots \oplus C_{i_s}^\circ.$$

We conclude in particular: if $\mathbf{v}_1^{i_j}, \ldots, \mathbf{v}_{k_{i_j}}^{i_j}$ are a basis for $C_{i_j}^\circ$ then $\{\mathbf{v}_l^{i_j} \mid l = 1, \ldots, k_{i_j}; j = 1, \ldots, s\}$ is a set of linearly independent vectors.

But if $C_{i_j} = \bigcap_l F_l^{i_j}$ where $F_l^{i_j}$ are all the facets of this agent's TH of demand which contain C_{i_j} in their boundary, then applying Lemma B.3.2 again, $C_{i_j}^\circ$ is the sum of the spaces $(F_l^{i_j})^\circ$. Each $(F_l^{i_j})^\circ$ is spanned by a single vector $\mathbf{v}_l^{i_j}$, which we may choose to be a primitive integer vector. We may select a maximal linearly independent subset of these vectors, and re-index so these are $\{\mathbf{v}_l^{i_j} \mid l = 1, \ldots, k_{i_j}\}$. Then $C_{i_j}^\circ = \bigoplus_{l=1}^{k_{i_j}} (F_l^{i_j})^\circ$. We already know that $C_{i_j} \subseteq \bigcap_{l=1}^{k_{i_j}} F_l^{i_j}$ so it follows (by Lemma B.3.2) that the affine spans of C_{i_j} and $\bigcap_{l=1}^{k_{i_j}} F_l^{i_j}$ coincide. It follows that C_{i_j} is given, locally around $\mathbf{p}$, by the intersection of the facets $F_1^{i_j}, \ldots, F_{k_{i_j}}^{i_j}$; these facets were chosen above such that their normal vectors are linearly independent. □

We now have the technical results we need to prove Theorem 5.6.

Proof of Theorem 5.6 Proposition 5.9 covers the case in which condition of the theorem is not satisfied. So suppose that the condition is satisfied. Suppose we m agents and for $j = 1, \ldots, m$ their valuation is $u^j : A_j \to \mathbb{R}$; write $U : A \to \mathbb{R}$ for the aggregate valuation (as in Section 5.1). We have the tropical variety $\mathcal{T}_U$ of aggregate demand, and the corresponding SNP.

This SNP provides a subdivision of $\mathrm{Conv}(A)$. Our bundle $\mathbf{x}$ may lie at a vertex of the subdivision, in which case there exists a price vector at which it is uniquely demanded. If not, it lies in some k-face of the SNP for some $k \neq 0$. Let $\Delta_{\mathbf{x}}$ be one such k-face. Let $\mathbf{p}_{\mathbf{x}} \in \mathbb{R}^n$ be a price in the corresponding $(n-k)$-cell $C_{\mathbf{x}}$ of aggregate demand. The set $\{\mathbf{y}^\beta \mid \beta \in B\}$ of vertices of $\Delta_{\mathbf{x}}$ are the bundles which are uniquely demanded in an open (n-dimensional) region of $\mathbb{R}^n$ with $C_{\mathbf{x}}$ in its boundary. By assumption there exist $\lambda_\beta \in [0, 1]$ with $\sum_\beta \lambda_\beta = 1$ such that $\mathbf{x} = \sum_\beta \lambda_\beta \mathbf{y}^\beta$.

Suppose that $\mathbf{x}$ is not demanded on aggregate at any price. Then, as in the proof of Lemma 3.1, it must follows that $U(\mathbf{x}) < \sum_\beta \lambda_\beta U(\mathbf{y}^\beta)$.

Pick ϵ so that

$$U(\mathbf{x}) < \sum_\beta \lambda_\beta U(\mathbf{y}^\beta) - \epsilon.$$

Now apply Lemma B.5, perturbing agents $j = 2, \ldots, m$ so that their valuation function is altered by no more than $\frac{\epsilon}{3m}$, where we recall that m is the number of agents present. It follows, by assumption regarding the demand type $\mathcal{D}$, that the conditions of Proposition 5.10 are satisfied at any intersection of agents' demands. Let U' be the new aggregate demand.

Now $\mathbf{x}$ lies in some k-face of the SNP of this new aggregate demand U', which corresponds to some $(n-k)$-cell of $\mathcal{T}_{U'}$. Let price $\mathbf{p}' \in \mathbb{R}^n$ be in this cell. By Proposition 5.10, it follows that $\mathbf{x} \in D_{U'}(\mathbf{p}')$.

However, $\mathbf{x} \in D_{U'}(\mathbf{p}')$ means that $\mathbf{x}$ is weakly preferred on aggregate to any other bundle – including all those in our original vertex set $\{\mathbf{y}^\beta\}$. So, for each $\beta \in B$, we have

$$U'(\mathbf{x}) - \mathbf{x}.\mathbf{p}' \geq U'(\mathbf{y}^\beta) - \mathbf{y}^\beta.\mathbf{p}'. \tag{5}$$

But $U'(\mathbf{x}) = \sum_{j=1}^m (u^j)'(\mathbf{x}^j)$, where $\mathbf{x}^j \in A_j$ is the bundle accorded to agent j under this optimal allocation (in particular $\sum_j \mathbf{x}^j = \mathbf{x}$) and $(u^j)'$ is the agent's perturbed valuation function. So

$$\|U'(\mathbf{x}) - U(\mathbf{x})\| = \|\sum_{j=1}^{m}[(u^j)'(\mathbf{x}^j) - u^j(\mathbf{x}^j)]\| \leq \sum_{j=1}^{m}\|(u^j)'(\mathbf{x}^j) - u^j(\mathbf{x}^j)\| \leq m \cdot \frac{\epsilon}{3m} = \frac{\epsilon}{3}$$

and hence $U(\mathbf{x}) + \frac{\epsilon}{3} \geq U'(\mathbf{x})$. Similarly, for all $\beta \in B$, we have $\|U'(\mathbf{y}^\beta) - U(\mathbf{y}^\beta)\| \leq \frac{\epsilon}{3}$ and so $U'(\mathbf{y}^\beta) \geq U(\mathbf{y}^\beta) - \frac{\epsilon}{3}$. Putting these facts together in line (5) we find:

$$U(\mathbf{x}) - \mathbf{x}.\mathbf{p}' \geq U(\mathbf{y}^\beta) - \mathbf{y}^\beta.\mathbf{p}' - \frac{2\epsilon}{3}.$$

Since this holds for all vertices $\mathbf{y}^\beta$ of our original k-face $\Delta_\mathbf{x}$ of the SNP, it follows that we may take a weighted sum, using the same weights as originally identified:

$$U(\mathbf{x}) - \mathbf{x}.\mathbf{p}' \geq \sum_\beta \lambda_\beta U(\mathbf{y}^\beta) - \sum_\beta \lambda_\beta \mathbf{y}^\beta.\mathbf{p}' - \frac{2\epsilon}{3} \implies U(\mathbf{x}) \geq \sum_\beta \lambda_\beta U(\mathbf{y}^\beta) - \frac{2\epsilon}{3}.$$

But we originally chose ϵ to satisfy $U(\mathbf{x}) < \sum_\beta \lambda_\beta U(\mathbf{y}^\beta) - \epsilon$. This contradiction completes the proof. □

B.4 Proof of results in Section 5.3

Proof of Theorem 5.16. Consider a 3-dimensional demand type $\mathcal{D}$ whose vectors span $\mathbb{R}^3$, and which always has competitive equilibrium. By a basis change, we can assume $\mathcal{D}$ includes the unit coordinate vectors, $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$,[55] so any coordinate entry in any $\mathbf{v} \in \mathcal{D}$ must be 0 or ± 1 (since if not, there is a determinant of $\mathbf{v}$ together with two of $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ that exceeds 1 in magnitude).

Now if $\mathcal{D}$ contains a vector $\mathbf{w}$ with non-zero entry in each coordinate direction, then by a basis change, exchanging any $\mathbf{e}^i$ with $-\mathbf{e}^i$ as necessary, we can assume that $\mathbf{w} = \mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3$. We now consider the two cases $\mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3 \in \mathcal{D}$, and $\mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3 \notin \mathcal{D}$, in turn.

In either case, however, $\mathcal{D}$ cannot contain all three of $\mathbf{e}^1 + \mathbf{e}^2$, $\mathbf{e}^2 + \mathbf{e}^3$ and $\mathbf{e}^1 + \mathbf{e}^3$, since

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2.$$

So suppose w.l.o.g. that $\mathbf{e}^1 + \mathbf{e}^3 \notin \mathcal{D}$ in both cases.

If $\mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3 \in \mathcal{D}$, then $\mathbf{e}^1 - \mathbf{e}^2 \notin \mathcal{D}$, since

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -2,$$

[55] There exist $\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3 \in \mathcal{D}$ spanning $\mathbb{R}^3$; it follows that matrix G with these vectors as columns is unimodular. Since $G^{-1}\mathbf{v}^i = \mathbf{e}^i$ for $i = 1, 2, 3$, demand of type $G^{-1}\mathcal{D}$ contains the unit coordinate vectors.

and likewise $\mathbf{e}^2 - \mathbf{e}^3 \notin \mathcal{D}$, $\mathbf{e}^1 - \mathbf{e}^3 \notin \mathcal{D}$. So all the vectors of $\mathcal{D}$ must be columns of

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

(although not all these columns *need* be in $\mathcal{D}$). Pre-multiplying this matrix by the unimodular matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

yields a matrix all of whose columns are in $\mathcal{D}^3_{ss}$, so $\mathcal{D}$ is a basis change of a subset of $\mathcal{D}^3_{ss}$.

If, instead, $\mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3 \notin \mathcal{D}$, and if $\mathcal{D}$ is not a subset of $\mathcal{D}^3_{ss}$ (in which case we are done), then $\mathcal{D}$ must contain at least one of $\mathbf{e}^1 + \mathbf{e}^2$ and $\mathbf{e}^2 + \mathbf{e}^3$ (since we already have $\mathbf{e}^1 + \mathbf{e}^3 \notin \mathcal{D}$). So assume w.l.o.g. that $\mathbf{e}^1 + \mathbf{e}^2 \in \mathcal{D}$. Then $\mathbf{e}^1 - \mathbf{e}^2 \notin \mathcal{D}$, since

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -2$$

and likewise only one of $\mathbf{e}^2 + \mathbf{e}^3$ and $\mathbf{e}^2 - \mathbf{e}^3$ can be in $\mathcal{D}$. Also, if $\mathbf{e}^2 - \mathbf{e}^3 \in \mathcal{D}$ then $\mathbf{e}^1 - \mathbf{e}^3 \notin \mathcal{D}$, since

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{vmatrix} = -2.$$

So if $\mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3 \notin \mathcal{D}$, and if $\mathcal{D}$ is not a subset of $\mathcal{D}^3_{ss}$, the vectors of $\mathcal{D}$ must all be contained in the columns of just one of the two matrices

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

But pre-multiplying these two matrices by the two unimodular matrices,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively, yields matrices all of whose columns are in $\mathcal{D}^3_{ss}$.

So if equilibrium always exists for $\mathcal{D}$, and the vectors of $\mathcal{D}$ span $\mathbb{R}^3$ then $\mathcal{D}$ is a basis change from a subset of $\mathcal{D}^3_{ss}$.

Suppose next that the vectors for $\mathcal{D}$ do not span $\mathbb{R}^3$. As before we may perform a basis change of $\mathcal{D}$ so that this time $\mathbf{e}^1$ and $\mathbf{e}^2 \in \mathcal{D}$. Since the span of the vectors in $\mathcal{D}$ now has dimension 2, we conclude that (after the aforementioned basis change) all vectors in $\mathcal{D}$ have 0 third coordinate; as before, their first and second coordinates may only be ± 1 or 0. So additional vectors in $\mathcal{D}$ can only be $\mathbf{e}^1 - \mathbf{e}^2$ or $\mathbf{e}^1 + \mathbf{e}^2$. These vectors cannot both be in $\mathcal{D}$.[56] In the former case we have $\mathcal{D} \subsetneq \mathcal{D}^3_{ss}$, and in the latter $\mathcal{D}$ is transformed to a subset of $\mathcal{D}^3_{ss}$ after basis change by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The fact that equilibrium always exists for any basis change from a subset of $\mathcal{D}^3_{ss}$ follows immediately from the discussion in Section 5.3.2 and Proposition 5.5. $\square$

Checking that the 4-D example of Section 5.3.3 always has a competitive equilibrium

We check that $\mathcal{D}$ does satisfy the criterion of Corollary 5.7 using Matlab. [Further details to be completed.]

References

L. M. Ausubel and P. Milgrom. Ascending auctions with package bidding. *Frontiers of Theoretical Economics*, 1(1):1–42, 2002.

E. Baldwin and P. Klemperer. The definition of substitutes for indivisible goods. University of Oxford, Department of Economics discussion paper, 2012. Available from author websites `http://users.ox.ac.uk/~wadh1180` and `www.paulklemperer.org`

E. Baldwin and P. Klemperer. The multi-dimensional product-mix auction and tropical geometry. In preparation.

S. Bikhchandani and J. Mamer. Competitive equilibrium in an exchange economy with indivisibilities. *Journal of Economic Theory*, 74(2):385–413, 1997.

J. W. S. Cassels. *An introduction to the geometry of numbers.* Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd. 99 Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959.

P. Cramton, Y. Shoham, and R. Steinberg. *Introduction to combinatorial auctions.* MIT Press, 2006.

S. Fujishige and Z. Yang. A note on Kelso and Crawford's gross substitutes condition. *Mathematics of Operations Research*, 28(3):463–469, 2003.

J. K. Goeree and A. Kushnir. A geometric approach to mechanism design. Working Paper 56, University of Zurich, Department of Economics, December 2011.

B. Grünbaum. *Convex polytopes*, volume XVI of *Pure and Applied Maths.* Interscience Publishers, 1st edition, 1967.

F. Gul and E. Stacchetti. Walrasian equilibrium with gross substitutes. *Journal of Economic Theory*, 87(1):95–124, 1999.

[56] The greatest common divisor of the determinant of any 2×2 matrices formed from them is 2; see Remark 5.8.3.

F. Gul and E. Stacchetti. The English auction with differentiated commodities. *Journal of Economic Theory*, 92(1):66 – 95, 2000.

J. W. Hatfield and P. R. Milgrom. Matching with contracts. *The American Economic Review*, 95(4):913–935, 2005.

J. W. Hatfield, S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp. Stability and competitive equilibrium in trading networks. Mimeo, Stanford Graduate School of Business, May 2011.

J. W. Hatfield, S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp. Stability and competitive equilibrium in trading networks. Mimeo, Stanford Graduate School of Business, May 2012.

A. S. Kelso and V. P. Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica*, 50(6):1483–1504, 1982.

P. Klemperer. A new auction for substitutes: Central bank liquidity auctions, the U.S. TARP, and variable product-mix auctions. Working paper, Oxford University, 2008.

P. Klemperer. The product-mix auction: A new auction design for differentiated goods. *Journal of the European Economic Association*, 8(2-3):526–536, 2010.

D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry*. draft available from `http://www.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.pdf`, Forthcoming.

G. Mikhalkin. Decomposition into pairs-of-pants for complex algebraic hypersurfaces. *Topology*, 43(5):1035–1065, 2004.

G. Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *Journal of the American Mathematical Society*, 18(2):313–377, 2005.

P. Milgrom. Putting Auction Theory to Work: The Simultaneous Ascending Auction. *Journal of Political Economy*, 108(2):245–272, 2000.

P. Milgrom. Assignment messages and exchanges. *American Economic Journal: Microeconomics*, 1(2):95–113, 2009.

P. Milgrom and B. Strulovici. Substitute goods, auctions, and equilibrium. *Journal of Economic Theory*, 144(1):212–247, 2009.

A. Milnes. Creating confidence in cash. *Blueprint*, October:14, 2010.

B. Osserman and S. Payne. Lifting tropical intersections. *Arxiv preprint arXiv:1007.1314*, 2010.

J. Richter-Gebert, B. Sturmfels, and T. Theobald. First steps in tropical geometry. In V. M. G.L. Litvinov, editor, *Idempotent Mathematics and Mathematical Physics*, volume 377 of *Contemporary Mathematics*, pages 289–317. American Mathematical Society, 2005.

L. Spence, A. Insel, and S. Friedberg. *Elementary linear algebra*. Prentice Hall, 2 edition, 2007.

N. Sun and Z. Yang. Equilibria and indivisibilities: Gross substitutes and complements. *Econometrica*, 74(5):1385–1402, 2006.

N. Sun and Z. Yang. A double-track adjustment process for discrete markets with substitutes and complements. *Econometrica*, 77(3):933–952, 05 2009.

A. Teytelboym. Gross substitutes and complements: a simple generalisation. In preparation, 2012.